UNIVERSIDADE FEDERAL DO PARANÁ

RICARDO PALEARI DA SILVA

GLOBAL ANALYTIC HYPOELLIPTICITY FOR A CLASS OF LEFT-INVARIANT OPERATORS ON $\mathbb{T}^1 \times \mathbb{S}^3$

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FEVEREIRO 2020

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Orientador: Prof Dr. Alexandre Kirilov

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Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em MATEMÁTICA da Universidade Federal do Paraná foram convocados para realizar a arguição da tese de Doutorado de **RICARDO PALEARI DA SILVA** intitulada: **Global Analytic Hypoellipticity for a Class of Left-invariant Operators on $T^1 \times S^3$**, sob orientação do Prof. Dr. ALEXANDRE KIRILOV, que após terem inquirido o aluno e realizada a avaliação do trabalho, são de parecer pela sua APROVADO no rito de defesa.

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RESUMO

Apresentamos uma caracterização completa da hipoeliticidade global analítica de uma classe de operadores de primeira ordem definidos em alguns produtos de grupos de Lie compactos, principalmente $\mathbb{T}^1 \times \mathbb{S}^3$. No caso de coeficientes com valores reais, provamos que o operador é conjugado a um operador com coeficientes constantes e que tal conjugação preserva a hipoeliticidade global analítica. No caso em que a parte imaginária não é identicamente nula, nós mostramos que o operador é globalmente analítico hipoelítico se a condição $(\mathcal{P})$ de Nirenberg-Treves vale em conjunto com uma condição Diofantina. Também estendemos parte de nossos resultados para uma classe de operadores definidos em produtos da forma $\mathbb{T}^1 \times \mathbb{S}^3 \times \cdots \times \mathbb{S}^3$.

Palavras-chaves: Hipoeliticidade Global Analítica. Séries de Fourier em grupos de Lie compactos. Condições Diofantinas analíticas. Condição $(\mathcal{P})$ de Nirenberg-Treves.

ABSTRACT

We present a complete characterization to the global analytic hypoellipticity of a class of first-order operators defined on some products of compact Lie groups, mainly $\mathbb{T}^1 \times \mathbb{S}^3$. In the case of real-valued coefficients, we prove that the operator is conjugated to a constant coefficient operator and that such conjugation preserves the global analytic hypoellipticity. In the case where the imaginary part of the coefficients is not identically zero, we show that the operator is globally analytic hypoelliptic if the Nirenberg-Treves condition $(\mathcal{P})$ holds in addition to a Diophantine condition. We also extend part of our results for a class of operators defined on products of the type $\mathbb{T}^1 \times \mathbb{S}^3 \times \cdots \times \mathbb{S}^3$.

Keywords: Global analytic hypoellipticity. Fourier Series on compact Lie groups. analytic Diophantine conditions. Nirenberg-Treves condition $(\mathcal{P})$.

Contents

Introduction

Since the 1970s, the property called Global Hypoellipticity is being studied for different classes of (pseudo-)differential operators defined on different manifolds. The pioneering work in this area is the paper [17] of S. Greenfield and N. Wallach, which relates the global hypoellipticity of constant-coefficient vector fields defined on tori with the growth of the operator's symbol at infinity. In particular, on the 2-torus, this property translates into a Diophantine condition, that is, this study of this property becomes a problem on approximation by rational numbers.

This leads us to one of the major problems in this area, the Greenfield's and Wallach's conjecture [18], which claims the following: if a vector field defined on a closed connected orientable manifold is globally hypoelliptic, then the manifold is diffeomorphic to a torus and the vector field is conjugated to a Diophantine constant vector field. There are positive partial answers for this conjecture, for example, it is true in dimensions 2 and 3 ([21] and [16]).

Then, a lot of different directions are natural to consider. For example, one can consider systems of partial differential equations; classes of pseudo-differential operators; or more general classes of regularity, like the classes of Gevrey and Komatsu. In this work, we are interested in investigating the property called Global Analytic Hypoellipticity for a class of operators defined on compact Lie Groups.

Our choice is natural in the sense that the standard approach used by Greenfield and Wallach is based in Fourier analysis in tori; and on compact Lie Groups, there is a very well established Fourier theory.

The sense of an operator being globally hypoelliptic or being globally analytic hypoelliptic in this work are the following.

Definition 0.1. *Let G be a compact Lie group. A linear operator $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is called globally hypoelliptic (GH) if the following condition is true:*

$$u \in \mathcal{D}'(G) \text{ and } P(u) \in C^\infty(G) \Rightarrow u \in C^\infty(G).$$

Similarly, we say that P is globally analytic hypoelliptic (GAH) if the following condition is true:

$$u \in \mathcal{D}'(G) \text{ and } P(u) \in C^\omega(G) \Rightarrow u \in C^\omega(G).$$

In order to give a friendly version of our results at this moment, avoiding concepts that will be carefully introduced in the first two chapters of this work, we will state our results in $\mathbb{T}^1 \times \mathbb{S}^3$. However, we note that, under certain conditions, it is possible to obtain more general versions.

We start by considering first-order operators of form

$$P = \partial_t + (a + ib)(t)\partial_0 + q, \tag{1}$$

where $q \in \mathbb{C}$, $a, b : \mathbb{T}^1 \to \mathbb{R}$ are real-analytic functions and ∂_0 is the left-invariant vector field on $\mathbb{S}^3$ known as the neutral operator.

Global analytic (and smooth) hypoellipticity of vector fields and system of vector fields has been extensively studied in tori, within which we cite as most important and inspiring for this project [2, 3, 4, 7, 12, 17, 20, 21].

In the specific case of global analytic hypoellipticity on the torus $\mathbb{T}^2$, Bergamasco proved in [2] that $\partial_t + (a(t) + ib(t))\partial_x$, is (GAH) if and only if either $b(t)$ does not change sign or $b \equiv 0$ and the real number $a_0 = \frac{1}{2\pi}\int_0^{2\pi} a(t)dt$ is neither rational nor an exponential-Liouville number.

Recall that an irrational number λ is said to be an exponential Liouville number if there exists $\epsilon > 0$ such that the inequality $|\lambda - p/q| \leq e^{-\epsilon q}$ has infinitely many rational solutions p/q.

Next, in [8], Bergamasco and Zani proved that, if there exists a non-singular, globally analytic hypoelliptic vector field L on a compact surface M, then M is real analytically diffeomorphic to $\mathbb{T}^2$ and, either, the Nirenberg-Treves condition ($\mathcal{P}$) holds in M, or there are coordinates on which we can write $L = g(x,t)(\partial_t + \lambda\partial_x)$, where $g \neq 0$ everywhere and λ is a real number which is neither rational nor exponential-Liouville.

Our results in $\mathbb{T}^1 \times \mathbb{S}^3$ recover part of the behavior identified by Bergamasco and Zani in dimension 2, involving the Nirenberg-Treves condition ($\mathcal{P}$) and an analytic Diophantine condition, suggesting the existence of a real analytic version of the famous Greenfield's and Wallach's conjecture, see [16].

In the case of constant-coefficients operators, our main result is as follows.

Theorem 0.2. *Let $c, q \in \mathbb{C}$. The operator $L = \partial_t + c\partial_0 + q$ is globally analytic hypoelliptic if and only if the following condition holds: for all $B > 0$, there is $K_B > 0$ such that for all $k, \ell \in \mathbb{Z}$,*

$$\left|k + \tfrac{1}{2}\ell c - iq\right| \geq K_B e^{-B(|k|+|\ell|)}. \tag{ADC3}$$

For example, writing $c = a + ib$, with $a, b \in \mathbb{R}$, if $b \neq 0$ and $\mathrm{Re}(q)/b \notin \frac{1}{2}\mathbb{Z}$, then L is (GAH). And when $b = 0$ and $iq \in \mathbb{Z}$, then L is (GAH) if and only if a is neither rational nor exponential-Liouville.

This theorem follows directly from Propositions 3.1 and 3.5, Remark 3.8 and Lemma 3.9, while the details of the above example are given in Example 3.10.

For the general case (1), we introduce the following notation:

$$P_0 \doteq \partial_t + (a_0 + ib_0)\partial_0 + q,$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} a(s)ds \ \text{ and } \ b_0 = \frac{1}{2\pi}\int_0^{2\pi} b(s)ds.$$

Theorem 0.3. *The operator $P = \partial_t + (a + ib)(t)\partial_0 + q$ is (GAH) if and only if one of the following conditions holds:*

1. *if $b \not\equiv 0$ then b does not change sign; and either*

$$\frac{\mathrm{Re}(q)}{b_0} \notin \tfrac{1}{2}\mathbb{Z} \ \text{ or } \ \mathrm{Im}(q) + \mathrm{Re}(q)\frac{a_0}{b_0} \notin \mathbb{Z}.$$

2. *if $b \equiv 0$, then the condition (ADC3) holds; and either*

$$\mathrm{Re}(q) \neq 0 \ \text{ or } \ \mathrm{Im}(q) \notin \mathbb{Z} + \frac{a_0}{2}\mathbb{Z}.$$

The previous theorem is a consequence of Proposition 4.5 and Theorems 4.3, 4.10 and 4.11. Part of our proofs follows the ideas for the smooth case used in [4, 12, 13] which relies heavily on the use of cut-off functions. One of the difficulties of adapting such arguments is that in the analytic case there are no such functions. To overcome this problem we draw on ideas used by Bergamasco, Nunes and Zani in [6] to construct a singular solution, and results of Sjostrand, see [26], about the asymptotic behavior at the infinity of sequence of integrals involving analytic functions. Let us add one more remark about the theorem we just stated. It is not exactly the same one as the reader will find later in this thesis, but another formulation of it.

It is easy to see that the algebraic condition $\mathcal{N}_0 = \varnothing$ together with the hypothesis $b_0 \neq 0$ imply the so called (ADC) condition (see Example 3.10), so Theorem 0.3 will hold.

Finally, this thesis is organized as follows. Chapters 1 and 2 are dedicated to preliminary definitions and results. In Chapter 1 we discuss how the Fourier theory is defined on compact groups. This is based essentially on representation theory. We state the main definitions and results that will be used in this thesis. In Chapter 2 we discuss the particular case of the Lie group $SU(2)$. It has a very nice and well-understood representation theory, which allows us to explicit every calculation and do Fourier theory by hand. There is no contribution by the author on these two chapters, everything there can be found in the suggested literature.

Next, in Chapter 3, we characterize completely the global analytic hypoellipticity for first-order constant-coefficient operators defined on a product of compact Lie groups. We also obtain equivalent analytic Diophantine conditions that will be important in the next chapters and to construct examples. In Chapter 4 we study the class of invariant evolution operators defined by (1). We obtain necessary and sufficient conditions as announced in Theorem 0.3. Finally, in the last chapter, we extend the results to invariant evolution operators with more variables defined on $\mathbb{T}^1 \times \mathbb{S}^3 \times ... \times \mathbb{S}^3$.

Chapter 1

Preliminaries: Fourier Analysis on Compact Lie Groups

In this chapter, we introduce the basic definitions, notations and preliminary results necessary for the development of this thesis. In the first section, we introduce the basic notions of Representation Theory and define the most important types of representations we are concerned about in this thesis. We end this section with one of the most important results in that area, which is known as the Peter-Weyl's Theorem. The second section is shorter and defines the Fourier coefficients of functions and distributions and their Fourier series. The third section is devoted to state results that show how the rate of decay of Fourier coefficients classify functions and distributions. A very careful presentation of these concepts and complete proofs of all the results presented in the first three sections can be found in the references [15] (chapters 1 and 2) and [25] (chapters 7, 8 and 10). The last section of this chapter is dedicated to results concerning partial Fourier coefficients on products of compact Lie groups and the main reference for this part is [23].

1.1 Representations of Topological Groups

If G is a group, a representation of G is a pair (V, φ) consisting of a (complex) vector space V together with a group morphism $\varphi : G \to GL(V)$. Sometimes V is called a G-module. The dimension of the representation is the dimension of the vector space V. If $\dim V < \infty$, we will use the notation $d_\varphi := \dim V$. If V has an inner product, we say that (V, φ) is unitary if $\mathrm{Im}(\varphi) \subset \mathcal{U}(V) = \{T : V \to V; T^* = T^{-1}\}$. Some of the notations that are commonly used are $g \cdot v := \varphi(g)(v) =: \varphi_g(v)$. In this way, it is common to say that V itself is a representation, without mention the specific map φ, which is usually implicit. We also say that G acts on V. If $W \subset V$ is such that $g \cdot w \in W$ for all $w \in W$, we say that W is a G-invariant subspace (or a subrepresentation, or a G-submodule of V). Finally, we say that V is irreducible if its only G-invariant subspaces are $\{0\}$ and V itself.

Example 1.1. *Let G be a group and $V = \mathcal{F}(G)$ the vector space of complex-valued functions defined on G. Consider $\pi_L : G \to GL(\mathcal{F}(G))$ defined by*

$$((\pi_L)(g))(f)(x) := f(g^{-1} \cdot x),$$

for each $g, x \in G$ and $f \in \mathcal{F}(G)$. Similarly, we can consider $\pi_R : G \to GL(\mathcal{F}(G))$ defined by

$$(\pi_R(g))(f)(x) = f(x \cdot g)$$

for $g, x \in G$ and $f \in \mathcal{F}(G)$. It is easy to see that π_L and π_R are group morphisms, so $\mathcal{F}(G)$ is a representation of G in at least two different ways.

Example 1.2. *Let $V = \mathbb{C}$ with the standard inner product. In this case, we have $GL(\mathbb{C}) = \mathbb{C}^*$ and $U(\mathbb{C}) = U(1) = S^1 = \mathbb{T}^1$. For each $\xi \in G = \mathbb{R}^n$, define $e_\xi : \mathbb{R}^n \to S^1$ by $e_\xi(x) = e^{ix\cdot\xi}$. It is clear that the correspondence $\xi \mapsto e_\xi$ defines a unitary representation of $\mathbb{R}^n$ of dimension 1, in particular, irreducible. Similarly, if $\xi \in \mathbb{Z}^n$, then $e_\xi : \mathbb{T}^n \to S^1$ is well defined and of course is also a morphism of groups. Thus, the correspondence $\xi \mapsto e_\xi$ defines a unitary and one-dimensional representation of $\mathbb{T}^n$.*

Definition 1.3. *Let G be a group. If $\varphi : G \to GL(V)$ and $\psi : G \to GL(W)$ are representations of G, we define a morphism between (φ, V) and (ψ, W) as a linear map $A : V \to W$ such that for all $g \in G$, $v \in V$, we have $A(\varphi_g(v)) = \psi_g(A(v))$. In the short notation of action, this relation is just $A(g \cdot v) = g \cdot A(v)$, that is, A is a linear map that commutes with the actions of G on V and on W. We say that (φ, V) and (ψ, W) are equivalent (or isomorphic), if there exists such a map A which is also an isomorphism of vector spaces. In this case, we will use the notation $\varphi \sim \psi$.*

Proposition 1.4. *If the representations (φ, V) and (ψ, W) are irreducible and $A : V \to W$ is a morphism between these two representations, then $A = 0$ or A is an isomorphism.*

Proof. It is enough to observe that Ker$A$ and ImA are subrepresentations of V. □

Corollary 1.5 (Schur Lemma)**.** *If $\varphi : G \to GL(V)$ is an irreducible representation of finite dimension and $A : V \to V$ is a morphism of representations, then there exists $\lambda \in \mathbb{C}$ such that $A = \lambda \cdot \mathrm{Id}_V$.*

Proof. Since $\dim V < \infty$, A has some eigenvalue $\lambda \in \mathbb{C}$, so $A - \lambda \mathrm{Id}_V$ is not invertible. But $A - \lambda \mathrm{Id}_V$ is also a morphism of representations, so by the above proposition we must have $A - \lambda \mathrm{Id}_V = 0$. $\square$

Corollary 1.6. *Let G be an abelian group. If $\varphi : G \to GL(V)$ is an irreducible representation of finite dimension, then* $\dim V = 1$.

Proof. For each fixed $g \in G$, the map $\varphi_G : V \to V$ if a morphism of representations because

$$\varphi_g(h \cdot x) = \varphi_h(\varphi_g(x)) = \varphi_{gh}(x) = \varphi_{hg}(x) = h \cdot \varphi_g(x).$$

By Schur Lemma, there exists $c \in \mathbb{C}$ such that $\varphi_g = c\mathrm{Id}_V$, so if $v \in V$, $v \neq 0$, then $\varphi_g(v) = cv$. Hence, span$\{v\}$ is a subrepresentation of V. Since span$\{v\} \neq \{0\}$, we must have $V =$ span$\{v\}$. $\square$

By the above corollary, we have in particular that every irreducible unitary representation of $\mathbb{T}^n$ must be a morphism of groups $f : \mathbb{T}^n \to S^1$.

Theorem 1.7. *If $f : \mathbb{T}^n \to S^1 \subset \mathbb{C}$ is a group morphism such that $f \in L^1(\mathbb{T}^n)$, then there exists $\xi \in \mathbb{Z}^n$ with $f = e_\xi$.*

Proof. Suppose for now that $n = 1$. We can think the function f as a 2π-periodic function $f : \mathbb{R} \to S^1$. Since $f \not\equiv 0$, there exists $\lambda > 0$ such that $\Lambda := \int_0^\lambda f(\tau)d\tau \neq 0$. So, if $x \in \mathbb{T}^1$ is fixed, we have

$$f(x) = \Lambda^{-1} \int_0^\lambda f(x)f(\tau)d\tau = \Lambda^{-1} \int_0^\lambda f(x+\tau)d\tau = \Lambda^{-1} \int_x^{x+\lambda} f(\tau)d\tau.$$

From the above formula, it follows by induction that $f \in C^\infty(\mathbb{T}^1)$. Taking the derivative of f we get

$$f'(x) = \Lambda^{-1}f(x+\lambda) - \Lambda^{-1}f(x) = \Lambda^{-1}f(x).f(\lambda) - \Lambda^{-1}f(x) = (\Lambda^{-1}f(\lambda) - \Lambda^{-1})f(x).$$

So the function f satisfies the differential equation $f' = C_0 f$, where $C_0 = \Lambda^{-1}(f(\lambda) - 1) \in \mathbb{C}$. Hence, we must have $f(x) = f(0)e^{C_0 x}$ and since $|f(0)| = 1$, we have $|f(x)| = |e^{C_0 x}| = e^{\mathrm{Re}(C_0 x)}$. But $|f(x)| = 1$ for all $x \in \mathbb{R}$, so $\mathrm{Re}(C_0) = 0$, which implies $C_0 = 2\pi i\xi$ for some

$\xi \in \mathbb{R}$. Finally, since $f(x) = e^{i2\pi x \cdot \xi}$ is periodic, we conclude that $\xi \in \mathbb{Z}$. For the general case we write

$$f(x) = f(x_1 e_1 + ... + x_n e_n) = f(x_1 e_1) \cdot ... \cdot f(x_n e_n).$$

Each function $x \mapsto f(x \cdot e_j)$ is a group morphism $f_j : \mathbb{T}^1 \to S^1$. If f is integrable, then each function $f_j \in L^1(\mathbb{T}^1)$. We apply the already proved result for each j and conclude the proof of the Theorem. $\square$

Definition 1.8. *If G is a topological group, a representation $\varphi : G \to GL(V)$ is strongly continuous if for each $v \in V$ the map $\varphi_v : G \to V$, $g \mapsto \varphi_v(g) \doteq \varphi(g, v)$, is continuous. We say that φ is topologically irreducible if the only closed subrepresentations of V are $\{0\}$ and V itself. We say that φ is cyclic if there exists $v \in V$ such that $V = \overline{\text{span}(\varphi(G)(v))}$. Such a vector will be called a cyclic vector.*

The proof of the next Proposition can be done using standard arguments of Zorn's Lemma.

Proposition 1.9. *If $\varphi : G \to GL(V)$ is strongly continuous, then there exist a family $\{V_\lambda\}_{\lambda \in \Lambda}$ of cyclic subrepresentations of V such that $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$.*

It is a well known fact that every compact topological group G has a unique normalized Haar measure, that is, a measure μ that is bi-invariant (with respect to all left and right translations), invariant by inversion and $\mu(G) = 1$. In the case where G is a compact Lie group, this existence is much more easier to prove. The idea is as follows. First, take any left-invariant metric on G (take any inner product on its Lie algebra and spread it on the whole group using left-translations) and consider the associated (positive) left-invariant volume form. Then we do the standard average process, that is, we define a new metric now using right-translations and integrate the resulting function over G. It is easy to see that the resulting metric is bi-invariant. Finally, one just normalized this metric to get the Haar measure of G. All groups we are going to deal with in this thesis will be compact Lie groups. We will assume that any integral taken over any compact group G will be with respect to its Haar measure. Besides that, from now on, every representation $\varphi : G \to \mathcal{U}(\mathcal{H})$ will be unitary with $\mathcal{H}$ being a Hilbert space.

Lemma 1.10. *Let G be a compact group and $\varphi : G \to \mathcal{U}(\mathcal{H})$ be an unitary strongly continuous representation. Suppose that $w \in \mathcal{U}$ is a cyclic vector with $\|w\| = 1$. Then*

$$\langle u, v \rangle_\varphi := \int_G \langle \varphi(x)u, w \rangle_{\mathcal{H}} \langle w, \varphi(x)v \rangle_{\mathcal{H}} d\mu_G$$

defines another inner product in $\mathcal{H}$. Moreover, φ is also unitary with respect to this new inner product and $\|u\|_\varphi \leq \|u\|_\mathcal{H}$ for all $u \in \mathcal{H}$.

Proof. For each fixed $u \in \mathcal{H}$, consider the function $f_u(x) := \langle \varphi(x)u, w\rangle_\mathcal{H}$. Then,

$$\begin{aligned} |f_u(x) - f_u(y)| &= |\langle \varphi(x)u, w\rangle_\mathcal{H} - \langle \varphi(y)u, w\rangle_\mathcal{H}| \\ &= |\langle (\varphi(x) - \varphi(y))u, w\rangle_\mathcal{H}| \\ &\leq \|(\varphi(x) - \varphi(y))u\|_\mathcal{H}\|w\|_\mathcal{H} \\ &= \|(\varphi(x) - \varphi(y))u\|_\mathcal{H}. \end{aligned}$$

But $\|(\varphi(x) - \varphi(y))u\|_\mathcal{H} \to 0$ as $x \to y$ since φ is strongly continuous, so $f_u \in C^0(G)$. In this way, $\langle u, v\rangle_\varphi = \int_G f_u \overline{f_v}$ is well defined. It is clear that this pairing is bilinear, anti-hermitian and non-negative. Also, if $\langle u, u\rangle_\varphi = 0$, then since f_0 is continuous we must have $\langle \varphi(x)u, w\rangle_\mathcal{H} = 0$ for all $x \in G$. Since φ is unitary, this is equivalent to $\langle \varphi(x^{-1})w, u\rangle_\mathcal{H} = 0$ for all $x \in G$. Since w is cyclic, this implies that $u = 0$. So $\langle \cdot, \cdot\rangle_\varphi$ is an inner product. Now, if $u \in \mathcal{U}$, we have

$$\|u\|_\varphi^2 = \int_G |\langle \varphi(x)u, w\rangle|^2 \leq \int_G \|\varphi(x)u\|_\mathcal{H}^2 \underbrace{\|w\|_\mathcal{H}^2}_{=1} = \int_G \|u\|_\mathcal{H}^2 = \|u\|_\mathcal{H}^2,$$

so $\|u\|_\varphi \leq \|u\|_\mathcal{H}$. Finally, let us prove that φ is unitary with respect to $\langle \cdot, \cdot\rangle_\varphi$:

$$\begin{aligned} \langle u, \varphi^*(y), v\rangle_\varphi &= \langle \varphi(y)u, v\rangle_\varphi \\ &= \textstyle\int_G \langle \varphi(xy)u, w\rangle \langle w, \varphi(x)v\rangle d\mu(x) \\ &= \textstyle\int_G \langle \varphi(xy)u, w\rangle \langle w, \varphi(xy)\varphi(y^{-1})v\rangle d\mu(x) \\ &= \textstyle\int_G \langle \varphi(x)u, w\rangle \langle w, \varphi(x)(\varphi(y^{-1}v))\rangle d\mu_x \\ &= \langle u, \varphi(y^{-1})v\rangle_\varphi. \end{aligned}$$

$\square$

Lemma 1.11. *Let $\langle \cdot, \cdot\rangle_\varphi$ be the inner product defined in the above lemma. There exists a compact, self-adjoint, positive-definitive operator $A \in \mathcal{B}(\mathcal{H})$, which is also a morphism of representations that satisfies $\langle u, Av\rangle_\mathcal{H} = \langle u, v\rangle_\varphi$.*

Proof. For each fixed $v \in \mathcal{H}$, the map $F_v(u) := \langle u, v\rangle_\varphi$ is a continuous linear functional:

$$|F_v(u)| \leq \|u\|_\varphi\|v\|_\varphi \leq \|u\|_\mathcal{H}\|v\|_\mathcal{H},$$

that is, $\|F_v\|_\varphi \leq \|v\|_\varphi$. By Riesz's Lemma, there exists an unique $A(v) \in \mathcal{H}$ such that

$$F_v(u) = \langle u, A(v)\rangle_\mathcal{H} \text{ for all } u \in \mathcal{H}.$$

- A is bounded;

$$\|Av\|_{\mathcal{H}}^2 = \langle Av, Av\rangle_{\mathcal{H}} = \langle Av, v\rangle_{\varphi} \leq \|Av\|_{\varphi}\|v\|_{\varphi} \leq \|Av\|_{\mathcal{H}}\|v\|_{\mathcal{H}},$$

So, if $Av \neq 0$, then $\|Av\|_{\mathcal{H}} \leq \|v\|_{\mathcal{H}}$.

- A is self-adjoint;

$$\langle u, A^*v\rangle_{\mathcal{H}} = \langle Au, v\rangle_{\mathcal{H}} = \overline{\langle v, Au\rangle}_{\mathcal{H}} = \overline{\langle v, u\rangle}_{\varphi} = \langle u, v\rangle_{\varphi} = \langle u, Av\rangle_{\mathcal{H}},$$

so $A^* = A$.

- A is positive-definite; Just note that $\langle u, Au\rangle_{\mathcal{H}} = \langle u, u\rangle_{\varphi}$.

- A is a morphism of representations;

$$\begin{aligned}\langle u, A(\varphi(y)v)\rangle_{\mathcal{H}} &= \langle u, \varphi(y)v\rangle_{\varphi} \\ &= \langle \varphi^{-1}(y)u, v\rangle_{\varphi} \\ &= \langle \varphi^{-1}(y)u, Av\rangle_{\mathcal{H}} \\ &= \langle u, \varphi(y)Av\rangle_{\mathcal{H}},\end{aligned}$$

for all $u, v \in \mathcal{H}$, so $A \circ \varphi(y) = \varphi(y) \circ A$. for all $y \in G$.

- A is compact;

Let $B = \{u \in \mathcal{H}; \|u\|_{\mathcal{H}} \leq 1\}$, our goal is to prove that $\overline{A(B)} \subset \mathcal{H}$ is compact. Let (v_j) be a sequence on $A(B)$ and (u_j) a sequence on B such that $A(u_j) = v_j$ for all j. By Banach-Alaoglu's Theorem, B is weakly compact, so there exists a sub-sequence $(u_{j_k})_k$ which is weakly-convergent to some $u \in B$, that is, $\langle u_{j_k}, v\rangle_{\mathcal{H}} \to \langle u, v\rangle_{\mathcal{H}}$ for all $v \in \mathcal{H}$. Let us prove that $v_{j_k} \to Au$.

$$\begin{aligned}\|v_{j_k} - Au\|_{\mathcal{H}}^2 &= \|A(u_{j_k}) - Au\|_{\mathcal{H}}^2 = \langle A(u_{j_k} - u), A(u_{j_k} - u)\rangle_{\mathcal{H}} \\ &= \langle A(u_{j_k} - u), u_{j_k} - u\rangle_{\varphi} = \int_G g_k d\mu_G,\end{aligned}$$

where $g_k(x) = \langle \varphi(x)A(u_{j_k} - u), w\rangle_{\mathcal{H}}\langle w, \varphi(x)(u_{j_k} - u)\rangle_{\mathcal{H}}$. Note that

$$\begin{aligned}|g_k(x)| &= |\langle \varphi(x)A(u_{j_k} - u), w\rangle_{\mathcal{H}}| \cdot |\langle w, \varphi(x)(u_{j_k} - u)\rangle_{\mathcal{H}}| \\ &= |\langle u_{j_k} - u, A\varphi(x^{-1})w\rangle_{\mathcal{H}}| \cdot |\langle \varphi(x^{-1})w, u_{j_k} - u\rangle_{\mathcal{H}}| \\ &\leq \|u_{j_k} - u\|_{\mathcal{H}} \underbrace{\|A\|_{\mathcal{H}}}_{\leq 1} \underbrace{\|w\|_{\mathcal{H}}}_{=1} \|u_{j_k} - u\|_{\mathcal{H}} \\ &\leq \|u_{j_k} - u\|_{\mathcal{H}}^2 \leq 2(\underbrace{\|u_{j_k}\|_{\mathcal{H}}^2}_{=1} + \underbrace{\|u\|_{\mathcal{H}}^2}_{=1}) = 4.\end{aligned}$$

So the sequence of functions $\{g_k\}_k$ is dominated by a constant function, which lives in $L^1(G)$ because $\mu(G) = 1 < \infty$. Besides that, $\langle u_{j_k} - u, v \rangle_{\mathcal{H}} \to 0$ for all $v \in \mathcal{H}$, so $g_k \to 0$ pointwise. Hence, we can apply Dominated Convergence Theorem and conclude that $\int_G g_k \to 0$, which implies $v_{j_k} \to Au$.

$\square$

Recall that if $A \in \mathcal{B}(\mathcal{H})$ is compact and self-adjoint, then

- $\sigma(A)$ is countable;
- $\dim \mathrm{Ker}(A - \lambda \mathrm{Id}) < \infty$ if $0 \neq \lambda \in \sigma(A)$;
- $\sigma(A) \setminus \{0\}$ is discrete;
- $\mathcal{H} = \bigoplus_{\lambda \in \sigma(A)} \mathrm{Ker}(A - \lambda \mathrm{Id})$.

Corollary 1.12. *Let G be a compact group and $\varphi : G \to \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation on a Hilbert space $\mathcal{H}$. Then there exists a decomposition of $\mathcal{H}$ in a direct sum (as representations) of a family of subrepresentations of $\mathcal{H}$ which are irreducible and finite dimensional.*

Proof. We already saw that every strongly continuous representation is a direct sum of cyclic representations, so we can suppose without loss of generality that φ is cyclic. Consider the operator A from the above lemma. Since A is positive-definite, we have $\mathrm{Ker}\, A = 0$, so $\dim(\mathrm{Ker} A - \lambda \mathrm{Id}) < \infty$ for all $\lambda \in \sigma(A)$ and $\mathcal{H} = \bigoplus_{\lambda \in \sigma(A)} (\mathrm{Ker} A - \lambda \mathrm{Id})$. Since A is a morphism of representations, each subspace $\mathrm{Ker} A - \lambda \mathrm{Id}$ is a subrepresentation. Besides that, each subspace $\mathrm{Ker} - \lambda \mathrm{Id}$ can also be decomposed as a sum of irreducible subrepresentations since they are finite dimensional, so is $\mathcal{H}$. $\square$

Corollary 1.13. *Irreducible strongly continuous unitary representations of compact groups are finite dimensional.*

Definition 1.14. *If G is compact, we denote by* $\mathrm{Rep}(G)$ *the set of all irreducible strongly continuous unitary representations of G. If G is just locally compact, in the definition of the set* $\mathrm{Rep}(G)$ *we just change the condition strongly continuous by continuous. The unitary dual of G is $\widehat{G} = \mathrm{Rep}(G)/\sim$, where we identify isomorphic representations.*

It is important to know that the set $\widehat{G}$ is countable if G is compact. This is a very well known fact and it has some generalizations, see for example [14].

Example 1.15. *We have that $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$. In fact, the correspondence $\mathbb{R}^n \to \widehat{\mathbb{R}^n}$, $\xi \mapsto [e_\xi]$, is a bijection. Similarly, we have $\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n$ since the correspondence $\mathbb{Z}^n \to \widehat{\mathbb{T}^n}$, $\xi \mapsto [e_\xi]$, is also a bijection.*

Proposition 1.16. *Let G be a compact group. If $\xi \in \widehat{G}$, with $\dim \xi = m$, then ξ has a matrix representative, that is, there exists a continuous morphism of groups $\varphi : G \to \mathcal{U}(m)$ such that $\xi = [\varphi]$.*

Proof. Let $\psi \in \xi$, $\psi : G \to \mathcal{U}(\mathcal{H})$, $\dim \mathcal{H} < \infty$, and fix an orthonormal basis $\beta = \{e_j\}_{j=1}^m$ of $\mathcal{H}$. Define, for each $x \in G$, the matrix

$$\varphi_{ij}(x) := \langle \psi(x)e_j, e_i \rangle, 1 \leq i, j \leq m,$$

which is the matrix that represents $\psi(x) \in \mathcal{U}(\mathcal{H})$ in the basis β, so $\varphi = (\varphi_{ij}) : G \to \mathcal{U}(m)$ is a unitary representation. Let us see that $\xi \sim \varphi$. If $\{\tilde{e}_j\}_{j=1}^m$ denoted the canonical basis of $\mathbb{C}^m$, let $A : \mathcal{H} \to \mathbb{C}^m$ the unique isomorphism of vector spaces such that $A(e_j) = \tilde{e}_j$ for all $j = 1, 2, ..., m$. If $v = \sum_j \lambda_j e_j \in \mathcal{H}$, then

$$\varphi(x)Av = \varphi(x) \sum_{j=1}^m \lambda_j \tilde{e}_j = \sum_{j=1}^m \lambda_j (\varphi(x)\tilde{e}_j) = \sum_{j=1}^m \lambda_j \sum_{i=1}^m \langle \psi(x)e_j, e_i \rangle \tilde{e}_i.$$

On the other hand,

$$\begin{aligned} A(\psi(x)v) &= A\left(\psi(x)\sum_{j=1}^m \lambda_j e_j\right) = A\left(\sum_{j=1}^m \lambda_j \psi(x) e_j\right) \\ &= A\left(\sum_{j=1}^m \lambda_j \sum_{i=1}^m \langle \psi(x)e_j, e_i\rangle e_i\right) \\ &= \sum_{i=1}^m \left(\sum_{j=1}^m \lambda_j \langle \psi(x)e_j, e_i\rangle \underbrace{Ae_i}_{=\tilde{e}_i}\right) = \varphi(x)Av. \end{aligned}$$

□

Lemma 1.17. *Let G be a compact group, $\xi, \eta \in \widehat{G}$, $\xi \ni \varphi = (\varphi_{ij})_{ij,=1}^m$ and $\eta \ni \psi = (\psi_{kl})_{k,l=1}^n$. Then,*

$$\langle \varphi_{ij}, \psi_{kl} \rangle_{L^2(G)} = \begin{cases} 0 & \text{if } \ \xi \neq \eta \\ \frac{1}{m}\delta_{ik}\delta_{jl} & \text{if } \ \xi = \eta \end{cases}.$$

Proof. Fix $1 \leq j \leq m$, $1 \leq l \leq n$, and define $E_{pq} = 1$ if $p = j, q = l$ and $E_{pq} = 0$ in any other case, with $E \in \mathbb{C}^{m \times n}$. We also define

$$A := \int_G \varphi(y) E \psi(y)^{-1} d\mu_G(y) \in \mathbb{C}^{m \times n}.$$

Note that A is a morphism of representations because:

$$\begin{aligned}
\varphi(x)A &= \int_G \varphi(x)\varphi(y) E \psi(y^{-1}) d\mu(y) \\
&= \int_G \varphi(xy) E \psi((xy)^{-1}x) d\mu(y) \\
&= \int_G \varphi(y) E \psi(y^{-1}x) d\mu(y) \\
&= \int_G \varphi(y) E \psi(y^{-1}) \psi(x) d\mu(y) = A\psi(x).
\end{aligned}$$

By Schur's Lemma, since φ and ψ are irreducible of finite dimension we must have

$$A = \begin{cases} 0 & \text{if} \quad \varphi \not\sim \psi \\ \lambda \cdot \mathrm{Id} & \text{if} \quad \varphi = \psi \end{cases}$$

for some $\lambda \in \mathbb{C}$. On the other hand,

$$\begin{aligned}
A_{ik} &= \int_G \sum_p \sum_q \varphi_{ip}(y) E_{pq} \psi_{qk}(y^{-1}) d\mu = \int_G \varphi_{ij}(y) \psi_{lk}(y^{-1}) d\mu \\
&= \int_G \varphi_{ij}(y) \psi_{lk}(y)^* d\mu = \langle \varphi_{ij}, \psi_{kl} \rangle_{L^2(G)}.
\end{aligned}$$

If $\varphi \not\sim \psi$, then $A = 0$, which implies $\langle \varphi_{ij}, \psi_{kl} \rangle = 0$ for all i, j, k, l. If $\varphi = \psi$, then $m = n$ and $\langle \varphi_{ij}, \varphi_{kl} \rangle = A_{ik} = 0$ if $i \neq k$ because A is diagonal. Now

$$\mathrm{tr}(A) = \int_G \mathrm{tr}(\varphi E \varphi^{-1}) d\mu = \int_G \mathrm{tr}(E) d\mu = \delta_{jl} = \lambda \cdot m.$$

hence

$$A_{ii} = \langle \varphi_{ij}, \varphi_{il} \rangle = \lambda = \frac{\delta_{jl}}{m}.$$

□

Recall the basic representations $\pi_L : G \to GL(\mathcal{F}(G))$, $\pi_R : G \to GL(\mathcal{F}(G))$, given by translations of functions. If $f \in L^2(G)$ and $g \in G$, then

$$\|\pi_L(g)(f)\|_2^2 = \int_G |f(g^{-1}x)|^2 d\mu = \int_G |f(x)|^2 d\mu = \|f\|_2^2,$$

so $L^2(G)$ is a G-invariant sub-space of $\mathcal{F}(G)$. Similarly, π_R also restricts to $L^2(G)$. The same calculation above also showed that these representations are unitary. In this way, we can consider the representations $\pi_L : G \to \mathcal{U}(L^2(G))$ and $\pi_R : G \to \mathcal{U}(L^2(G))$. Also, from now on, we will follow a certain convention. When one gives an element $\xi \in \widehat{G}$, we will choose once and for all a matrix representative $\varphi : G \to \mathcal{U}(d_\xi)$ in the class of ξ. The notation $\xi = [\varphi]$ will be used suggesting the notation of the already chosen matrix representative φ in that class.

Theorem 1.18 (Peter-Weyl)**.** *Let G be a compact group. Then*

$$\mathcal{B} = \{\sqrt{d_\varphi} \cdot \varphi_{ij}, \varphi : G \to \mathcal{U}(d_\varphi); [\varphi] \in \widehat{G}\},$$

is an orthonormal basis of $L^2(G)$. Moreover, for each $[\varphi] \in \widehat{G}$, we have

- $\mathcal{H}_i^\varphi :=$ *span*$\{\varphi_{ij}; 1 \leq j \leq d_\varphi\} \subset L^2(G)$ *is π_R-invariant,*
- $\varphi \sim \pi_R\Big|_{\mathcal{H}_i^\varphi}$ *;*
- $L^2(G) = \displaystyle\bigoplus_{[\varphi]\in\widehat{G}} \bigoplus_{i=1}^{d_\varphi} \mathcal{H}_i^\varphi$ *;*
- $\pi_R \sim \displaystyle\bigoplus_{[\varphi]\in\widehat{G}} \bigoplus_{i=1}^{d_\varphi} \varphi.$

Proof. Fix $1 \leq i \leq d_\varphi$. The fact that $\mathcal{H}_i^\varphi$ is π_R-invariant is direct.

- $\varphi \sim \pi_R|_{\mathcal{H}_i^\varphi}$;

  Let $\{e_i\}$ be the canonical basis of $\mathbb{C}^{d_\varphi}$, so $\varphi(y)e_j = \sum_k \varphi_{kj}(y)e_k$. Consider the linear isomorphism $A : \mathbb{C}^{d_\varphi} \to \mathcal{H}_i^\varphi$ such that $A(e_j) = \varphi_{ij}$ for all $j = 1, ..., d_\varphi$. Then A is a morphism between representations because

  $$\pi_R(y)A(e_j) = \pi_R(y)(\varphi_{ij}) = \sum_k \varphi_{kj}(y)\varphi_{ik},$$

  and

  $$A\varphi(y)(e_j) = A\left(\sum_k \varphi_{kj}(y)e_k\right) = \sum_k \varphi_{kj}(y)\varphi_{ik},$$

  so $\varphi \sim \pi_R|_{\mathcal{H}_i^\varphi}$.

- $\mathcal{B}$ is an orthonormal basis of $L^2(G)$;

  We already know that $\mathcal{B}$ is an orthonormal set. Let $\mathcal{H} := \bigoplus_{[\varphi]\in\widehat{G}} \bigoplus_{i=1}^{d_\varphi} \mathcal{H}_i^\varphi$ and suppose that $\mathcal{H} \neq L^2(G)$, so $\mathcal{H}^\perp \neq \{0\}$ is π_R-invariant. We know that $\pi_R|_{\mathcal{H}^\perp}$ is a sum of irreducible finite dimensional unitary representations, so there exists a no trivial $E \subset \mathcal{H}^\perp$ and a matrix unitary representation $\varphi = (\varphi_{ij})$ with $\varphi \sim \pi_R|_E$. Let $\{f_j\}_{i=1}^{d_E}$ be the orthonormal basis of E such that $\pi_R(y)f_j = \sum_i \varphi_{ij}(y)f_i$. This last equality happens in $L^2(G)$, in particular, for almost all $x \in G$ we have $f_j(xy) = \sum_i \varphi_{ij}(y)f_i(x)$. Consider now the sets

$$\begin{aligned} N(y) &= \{x \in G; f_j(xy) \neq \textstyle\sum_i \varphi_{ij}(y)f_i(x)\} \\ M(x) &= \{y \in G; f_j(xy) \neq \textstyle\sum_i \varphi_{ij}(y)f_i(x)\} \\ K &= \{(x,y) \in G \times G; f_j(xy) \neq \textstyle\sum_i \varphi_{ij}(y)f_i(x)\}, \end{aligned}$$

so we know that $\mu_G(N(y)) = 0$. Note that $N(y)$ is the y-section of K and $M(x)$ is the x-section of K. By Fubini's Theorem,

$$\mu_G(K) = \int_G \underbrace{\mu_G(M(x))}_{\geq 0} \, d\mu(x) = \int_G \underbrace{\mu(N(y))}_{=0} \, d\mu_G = 0,$$

so $\mu_G(M(x)) = 0$ for almost all $x \in G$. Let x_0 be a point such that $\mu_G(M(x_0)) = 0$, then

$$f_j(x_0 y) = \sum_i \varphi_{ij}(y) f_i(x_0)$$

holds for almost all $y \in G$. If $z = x_0 y$, we will have

$$\begin{aligned} f_j(z) &= \sum_i \varphi_{ij}(x_0^{-1}z)f_i(x_0) = \sum_i \sum_j \varphi_{ik}(x_0^{-1})\varphi_{kj}(z)f_i(x_0) \\ &= \sum_k \varphi_{kj}(z) \left(\underbrace{\sum_i \varphi_{ik}(x_0^{-1})f_i(x_0)}_{:=\lambda_k} \right), \end{aligned}$$

so $f_j(z) = \sum_k \lambda_k \varphi_{kj}(z)$ holds for almost all $z \in G$. Hence,

$$f_j \in \text{span}\{\varphi_{kj}; 1 \leq k \leq d_E\} \subset \bigoplus_{k=1}^{d_E} \mathcal{H}_k^\varphi \subset \mathcal{H},$$

so $E \subset \mathcal{H} \cap \mathcal{H}^\perp = \{0\}$, which is contradiction.

$\square$

Using the notation of the above Theorem, we will denote by $\mathcal{H}^\varphi$ the space $\bigoplus_{i=1}^{d_\varphi} \mathcal{H}_i^\varphi$.

1.2 Fourier Series and Trigonometric Polynomials

Here we introduce the trigonometric polynomials and finally define the Fourier coefficients of L^2 functions on compact groups.

Definition 1.19. *Let G be a compact group and $\mathcal{B} = \{\sqrt{d_\varphi} \cdot \varphi_{ij}; \varphi = (\varphi_{ij}), [\varphi] \in \widehat{G}\}$ the basis given by Peter-Weyl's Theorem. The space of trigonometric polynomials is defined by* $\text{Trig Pol}(G) = \text{span}(\mathcal{B})$.

Example 1.20. *If $f \in \text{Trig Pol}(\mathbb{T}^n)$, then*

$$f(x) = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi},$$

where $\widehat{f}(\xi) \neq 0$ at most on a finite number of $\xi \in \mathbb{Z}^n$.

Theorem 1.21. *The space* $\text{Trig Pol}(G)$ *is a dense sub-algebra in $C^0(G)$.*

The proof of this Theorem consists in to show that $\text{Trig Pol}(G)$ is an involutive subalgebra of $C^0(G)$ that separate points. Hence the result follows by Stone-Weierstrass Theorem.

Corollary 1.22. *In a compact group G, for all $f \in L^2(G)$ we can write*

$$f = \sum_{[\varphi] \in \widehat{G}} d_\varphi \sum_{i,j=1}^{d_\varphi} \langle f, \varphi_{ij} \rangle_{L^2(G)} \cdot \varphi_{ij},$$

which will be called the Fourier Series of f. Moreover, a Plancherel equality type holds

$$\|f\|_2^2 = \sum_{[\varphi] \in \widehat{G}} d_\varphi \sum_{i,j=1}^{d_\varphi} |\langle f, \varphi_{ij} \rangle_{L^2(G)}|^2.$$

Definition 1.23. *Let G be a compact group, $f \in L^1(G)$ and $\varphi = (\varphi_{ij})_{i,j=1}^{d_\varphi}$ with $[\varphi] \in \widehat{G}$. We define the φ-coefficient of the Fourier series of f as the matrix*

$$(\widehat{f}(\varphi))_{ij} := \int_G f(x) \varphi_{ij}^* d\mu_G = \langle f, \overline{\varphi_{ij}^*} \rangle = \langle f, \varphi_{ji} \rangle_{L^2(G)},$$

for $1 \leq i, j \leq d_\varphi$.

Proposition 1.24. *Let G be a compact group and $f \in L^2(G)$, then*

$$f(x) = \sum_{[\varphi] \in \widehat{G}} d_\varphi \text{tr} \left(\widehat{f}(\varphi) \cdot \varphi(x) \right),$$

and this series converges for almost all $x \in G$, and also in $L^2(G)$. The Plancherel's formula becomes

$$\|f\|_2^2 = \sum_{[\varphi] \in \widehat{G}} d_\varphi \text{tr} \left(\widehat{f}(\varphi) \widehat{f}(\varphi)^* \right).$$

Proof. Just note that

$$\mathrm{tr}(\widehat{f}(\varphi)\varphi(x)) = \sum_i \left(\widehat{f}(\varphi)\varphi(x)\right)_{ii} = \sum_{i,j} \widehat{f}(\varphi)_{ij}\varphi_{ji}(x) = \sum_{i,j} \langle f, \varphi_{ji}\rangle \varphi_{ji}(x),$$

which conclude the first part. For the second part, just recall that $\mathrm{tr}(AA^*) = \sum_{i,j} |A_{ij}|^2$. □

Example 1.25. *For $f \in L^2(\mathbb{T}^n)$, its Fourier Series is given by*

$$f(x) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)e^{i2\pi x\cdot k}$$

where $\widehat{f}(k) = \int_{\mathbb{T}^n} f(x)e^{-i2\pi x\cdot k}dx$.

1.3 Function Spaces

Now we present results about characterizations of smooth functions and distributions in terms of their Fourier coefficients. So, from now on, G will denote a compact Lie group. Recall that we already fixed a bi-invariant volume form on G, so that it defines the Haar measure on G. One particular operator that helps the characterization of smooth functions on G is the Laplacian operator. One way to define it is by taking the general Laplacian-Beltrami operator associated with the fixed bi-invariant Riemannian metric on G. A more constructive way is the following. If G is semi-simple, then the Laplace operator on G can be identified with its Casimir element $\Omega \in \mathcal{U}(G)$, where $\mathcal{U}(G)$ denotes the universal enveloping algebra of G. If G is not semisimple, one decomposes its Lie algebra as a direct sum of its semisimple part with its center, take a basis of the center $\{U_i\}_{i=1}^k$ of its Lie algebra, and consider $\mathcal{L} = \Omega + \sum_i U_i^2$. It can be proved that $\mathcal{L}$ coincides with the Laplacian-Beltrami operator of G (see pg 331 of [19]). The following Proposition is very important when one considers decay properties of Fourier coefficients.

Proposition 1.26. *For every $[\xi] \in \widehat{G}$, the space $\mathcal{H}^\xi$ is an eigenspace of $\mathcal{L}$ and $-\mathcal{L}|_{\mathcal{H}^\xi} = \lambda_\xi \mathrm{Id}$, for some $\lambda_\xi \geq 0$.*

Proof. Recall that by definition the operator $\mathcal{L}$ is bi-invariant, so it commutes with both $\pi_R(x)$ and $\pi_L(x)$ for all $x \in G$. By Peter-Weyl Theorem, it then commutes with all $\xi \in \widehat{G}$ (every representation in $\mathrm{Rep}(G)$ is the restriction of π_R to some finite-dimensional space). So $\mathcal{L}$ preserves both the spaces generated by lines and by columns of $\mathcal{H}^\xi$, and since the set of functions $\{\xi_{ij}\}$

is linearly independent, this implies that for each $1 \leq i, j \leq d_\xi$, we have $\mathcal{L}\xi_{ij} = c_{ij}\xi_{ij}$. Let us prove that all the constants c_{ij} are equal. We have

$$\begin{aligned}
\mathcal{L} \circ \pi_R(y)\xi_{ij}(x) &= \mathcal{L}(\xi_{ij}(xy)) \\
&= \mathcal{L}\left(\sum_{k=1}^{d_\xi} \xi_{ik}(x)\xi_{kj}(y)\right) \\
&= \sum_{k=1}^{d_\xi} c_{ik}\xi_{ik}(x)\xi_{kj}(y).
\end{aligned}$$

On the other hand,

$$(\pi_R(y) \circ \mathcal{L}\xi_{ij})(x) = c_{ij}\xi_{ij}(xy) = \sum_{k=1}^{d_\xi} c_{ij}\xi_{ik}(x)\xi_{kj}(y).$$

From orthogonality relations, we have $c_{ik}\xi_{kj}(y) = c_{ij}\xi_{kj}(y)$, which implies $c_{ik} = c_{ij}$ for all i, j, k. Now using the left-translation representation we can conclude similarly that $c_{kj} = c_{ij}$ for all i, j, k, so $c_{ij} = c$ for all i, j. It is a general fact that $\mathcal{L}$ is negative definite, so $-c =: \lambda_\xi \geq 0$. □

Suppose that G has dimension n and let $\{X_i\}_{i=1}^n$ be a basis of its Lie algebra. For a multi-index $\alpha = (\alpha_1, \cdots \alpha_m) \in \mathbb{N}_0^m$, we define the left-invariant differential operator of order $|\alpha|$

$$\partial^\alpha := Y_1 \cdots Y_{|\alpha|},$$

with $Y_j \in \{X_i\}_{i=1}^m$, $1 \leq j \leq |\alpha|$ and $\sum\limits_{j: Y_j = X_k} 1 = \alpha_k$ for every $1 \leq k \leq m$. It means that ∂^α is a composition of left-invariant derivatives with respect to the vectors $X_1, \ldots, X_n$ such that each X_k enters ∂^α exactly α_k times. We do not specify in the notation ∂^α the order of these vectors, but this will not be relevant in the arguments that we will use in this work.

Proposition 1.27. *The following statements are equivalent:*

(i) $f \in C^\infty(G)$*;*

(ii) $\partial^\alpha f \in C(G)$ *for all multi-index* α*;*

(iii) $(-\mathcal{L})^k f \in C(G)$ *for all* $k \in \mathbb{N}_0$*;*

(iv) $Lf \in C(G)$ *for all* $L \in \mathcal{U}(\mathfrak{g})$.

We equip $C^\infty(G)$ with the usual Fréchet space topology defined by the family of seminorms $p_\alpha(f) = \max_{x\in G} |\partial^\alpha f(x)|$. Thus, the convergence on $C^\infty(G)$ is just the uniform convergence of functions and all their derivatives: $f_k \to f$ in $C^\infty(G)$ if $\partial^\alpha f_k(x) \to \partial^\alpha f(x)$, for all $x \in G$, due to the compactness of G.

Definition 1.28. *We define the* **space of distributions** $\mathcal{D}'(G)$ *as the space of all continuous linear functionals on* $C^\infty(G)$ *with the usual notion of convergence: for* (u_j) *a sequence in* $\mathcal{D}'(G)$ *and* $u \in \mathcal{D}'(G)$*, we write* $u_j \to u$ *in* $\mathcal{D}'(G)$ *as* $j \to \infty$ *if* $u_j(\varphi) \to u(\varphi)$ *in* $\mathbb{C}$ *as* $j \to \infty$*, for all* $\varphi \in C^\infty(G)$*.*

For $u \in \mathcal{D}'(G)$ and $\varphi \in C^\infty(G)$, we write

$$\langle u, \varphi\rangle_G := u(\varphi).$$

If $u \in L^p(G)$, $1 \le p \le \infty$, we can identify u with a distribution in $\mathcal{D}'(G)$ (which we will still denote by u) in a canonical way by

$$\langle u, \varphi\rangle_G := \int_G u(x)\varphi(x)\, dx.$$

It can be proved that if $u_j \to u$ in $L^p(G)$, then $u_j \to u$ in $\mathcal{D}'(G)$. For $Y \in \mathfrak{g}$, we can differentiate $u \in \mathcal{D}'(G)$ with respect to the vector field Y:

$$\langle Yu, \varphi\rangle_G := -\langle u, Y\varphi\rangle_G,$$

for all $\varphi \in C^\infty(G)$. Similarly, for any multi-index α, we define

$$\langle \partial^\alpha u, \varphi\rangle_G := (-1)^{|\alpha|}\langle u, \partial^\alpha \varphi\rangle_G,$$

for all $\varphi \in C^\infty(G)$.

Definition 1.29. *The space* $\mathcal{M}(\widehat{G})$ *consists of all mappings*

$$F : \widehat{G} \to \bigcup_{[\xi]\in\widehat{G}} \text{End}(\mathcal{H}^\xi) \subset \bigcup_{m=1}^{\infty} \mathbb{C}^{m\times m}$$

satisfying $F([\xi]) \in \text{End}(\mathcal{H}_\xi)$*, for every* $\xi \in \widehat{G}$*. With respect to the matrix representations, we have* $F([\xi]) \in \mathbb{C}^{d_\xi \times d_\xi}$*.*

The space $L^2(\widehat{G})$ *consists of all mappings* $F \in \mathcal{M}(\widehat{G})$ *such that*

$$\|F\|^2_{L^2(\widehat{G})} := \sum_{[\xi]\in\widehat{G}} \dim(\xi)\|F([\xi])\|^2_{\mathtt{HS}} < \infty,$$

where

$$\|F([\xi])\|_{\mathtt{HS}} = \sqrt{\text{tr}(F([\xi])F([\xi])^*)}.$$

The space $L^2(\widehat{G})$ is a Hilbert space with the inner product

$$\langle E, F\rangle_{L^2(\widehat{G})} := \sum_{[\xi]\in\widehat{G}} \dim(\xi)\mathrm{tr}(E([\xi])F([\xi])^*).$$

Notice that for any $f \in L^2(G)$, we can define

$$\begin{array}{rccl} \widehat{f}: & \widehat{G} & \to & \bigcup_{m=1}^{\infty} \mathbb{C}^{m\times m} \\ & [\xi] & \mapsto & \widehat{f}(\xi), \end{array}$$

and by the Plancherel formula, we have $\widehat{f} \in L^2(\widehat{G})$. We have the Parseval's identity

$$\langle f, g\rangle_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \mathrm{tr}\left(\widehat{f}(\xi)\widehat{g}(\xi)^*\right) = \left\langle \widehat{f}, \widehat{g}\right\rangle_{L^2(\widehat{G})}.$$

Theorem 1.30. *The Fourier transform $f \mapsto \mathcal{F}_G f = \widehat{f}$ defines a surjective isometry from $L^2(G)$ to $L^2(\widehat{G})$. The inverse Fourier transform is given by*

$$\begin{aligned} (\mathcal{F}_G^{-1}H)(x) &= \sum_{[\xi]\in\widehat{G}} d_\xi\ \mathrm{tr}\left(H([\xi])\xi(x)\right) \\ &= \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{m,n=1}^{d_\xi} H([\xi])_{mn}\xi(x)_{nm} \end{aligned}$$

and we have

$$\mathcal{F}_G^{-1}\circ\mathcal{F}_G = \mathrm{Id}_{L^2(G)} \quad \textit{and} \quad \mathcal{F}_G\circ\mathcal{F}_G^{-1} = \mathrm{Id}_{L^2(\widehat{G})}.$$

Definition 1.31. *Let $u \in \mathcal{D}'(G)$ and $\xi = (\xi_{ij})_{i,j=1}^{d_\xi}$, $[\xi] \in \widehat{G}$. The ξ-**Fourier coefficient of** u is*

$$\widehat{u}(\xi) := \langle u, \xi^*\rangle_G \in \mathbb{C}^{d_\xi\times d_\xi},$$

that is,

$$\widehat{u}(\xi)_{ij} = \left\langle u, \overline{\xi_{ji}}\right\rangle_G.$$

For $u \in \mathcal{D}'(G)$, we have

$$u = \sum_{[\xi]\in\widehat{G}} d_\xi\ \mathrm{tr}(\widehat{u}(\xi)\xi) = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{i,j=1}^{d_\xi} \widehat{u}(\xi)_{ij}\xi_{ji},$$

where the convergence is in the distribution sense.

We define

$$\langle\xi\rangle := \sqrt{1+\lambda_{[\xi]}},$$

where $\lambda_{[\xi]} \geq 0$ is the eigenvalue of $-\mathcal{L}$ associated to the eigenspace $\mathcal{H}^\xi$ (See Theorem 1.26). These numbers will play a major role when characterizing functions and distributions via Fourier coefficients.

Proposition 1.32. *There exists $C > 0$ such that*

$$\lambda_{[\xi]} \leq \langle\xi\rangle^2 \leq C\lambda_{[\xi]},$$

for all non-trivial $[\xi] \in \widehat{G}$.

Proposition 1.33. *There exists a constant $C > 0$ such that the inequality*

$$\dim(\xi) \leq C\langle\xi\rangle^{\frac{\dim G}{2}}$$

holds for all $\xi \in \mathrm{Rep}(G)$.

Theorem 1.34. *The following statements are equivalent:*

(i) $f \in C^\infty(G)$*;*

(ii) for each $N > 0$, there exists $C_N > 0$ such that

$$\|\widehat{f}(\xi)\|_{\mathtt{HS}} \leq C_N\langle\xi\rangle^{-N},$$

for all $[\xi] \in \widehat{G}$;

(iii) for each $N > 0$, there exists $C_N > 0$ such that

$$|\widehat{f}(\xi)_{ij}| \leq C_N\langle\xi\rangle^{-N},$$

for all $[\xi] \in \widehat{G},\ 1 \leq i, j \leq d_\xi$.

Theorem 1.35. *The following statements are equivalent:*

(i) $u \in \mathcal{D}'(G)$*;*

(ii) there exist $C,\ N > 0$ such that

$$\|\widehat{u}(\xi)\|_{\mathtt{HS}} \leq C\langle\xi\rangle^{N},$$

for all $[\xi] \in \widehat{G}$;

(iii) there exist $C,\ N > 0$ such that

$$|\widehat{u}(\xi)_{ij}| \leq C\langle\xi\rangle^{N},$$

for all $[\xi] \in \widehat{G},\ 1 \leq i, j \leq d_\xi$.

Definition 1.36. *Let G be a compact Lie group and $A : C^\infty(G) \to C^\infty(G)$ be a continuous linear operator. We define the* **symbol of the operator** A **in** $x \in G$ **and** $\xi \in \mathrm{Rep}(G)$*, with* $\xi = (\xi_{ij})_{i,j=1}^{d_\xi}$*, as*

$$\sigma_A(x,\xi) := \xi(x)^*(A\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi},$$

where $(A\xi)(x)_{ij} := (A\xi_{ij})(x)$*, for all* $1 \leq i, j \leq d_\xi$.

For instance, if we take $A = -\mathcal{L}$, we get

$$\sigma_{\mathcal{L}}(x,\xi) = \xi(x)^*(-\mathcal{L}\xi)(x) = \xi(x)^*(\lambda_{[\xi]}\xi)(x) = \lambda_{[\xi]}\mathrm{Id}_{d_\xi}.$$

Theorem 1.37. *Let σ_A be the symbol of a continuous linear operator $A : C^\infty(G) \to C^\infty(G)$. Then,*

$$Af(x) = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\mathrm{tr}\left(\xi(x)^*\sigma_A(x,\xi)\widehat{f}(\xi)\right)$$

for every $f \in C^\infty(G)$ and $x \in G$.

Notice that the formula above is independent of the choice of the matrix representative in each class. Indeed, if $\xi \sim \psi$ are matrix representations, there exists a unitary matrix U such that $\xi(x) = U^*\psi(x)U$ for all $x \in G$. So $\widehat{f}(\xi) = U^*\psi(x)U$ and by the formula of the symbol of the operator A,

$$\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x) = (U^*\psi(x)^*U)(U^*A\psi U)(x)) = U^*\sigma_A(x,\psi)U.$$

Thus $\mathrm{tr}\left(\xi(x)^*\sigma_A(x,\xi)\widehat{f}(\xi)\right) = \mathrm{tr}\left(\psi(x)^*\sigma_A(x,\psi)\widehat{f}(\psi)\right)$, for all $x \in G$. In the particular case where $A : C^\infty(G) \to \mathbb{C}^\infty(G)$ is linear, continuous and left-invariant, that is, $A\pi_L(y) = \pi_L(y)A$, for all $y \in G$, we have that σ_A is independent of $x \in G$ and

$$\widehat{Af}(\xi) = \sigma_A(\xi)\widehat{f}(\xi),$$

for all $f \in C^\infty(G)$ and $[\xi] \in \widehat{G}$.

Proposition 1.38. *Let $A, B : C^\infty(G) \to C^\infty(G)$ be continuous linear operators and $\lambda \in \mathbb{C}$. Then for all $x \in G$ and $[\xi] \in \widehat{G}$ holds:*

1. $\sigma_{A+B}(x,\xi) = \sigma_A(x,\xi) + \sigma_B(x,\xi)$;

2. $\sigma_{\lambda A}(x,\xi) = \lambda\sigma_A(x,\xi)$;

3. *If B is a left-invariant operator, then* $\sigma_{AB}(x,\xi) = \sigma_A(x,\xi)\sigma_B(\xi)$.

Let $Y \in \mathfrak{g}$. Notice that iY is a left-invariant operator and

$$\begin{aligned}
\langle iYf, g\rangle_{L^2(G)} &= \int_G (iYf)(x)\overline{g(x)}\,dx \\
&= i\int_G \frac{d}{dt} f(x\exp(tY))\Big|_{t=0}\overline{g(x)}\,dx \\
&= i\frac{d}{dt}\int_G f(x\exp(tY))\overline{g(x)}\,dx\Big|_{t=0} \\
&= i\frac{d}{dt}\int_G f(x)\overline{g(x\exp(-tY))}\,dx\Big|_{t=0} \\
&= i\int_G f(x)\overline{\frac{d}{dt}g(x\exp(-tY))\Big|_{t=0}}\,dx \\
&= i\int_G f(x)\overline{(-Yg)(x)}\,dx \\
&= \int_G f(x)\overline{(iYg)(x)}\,dx \\
&= \langle f, iYg\rangle_{L^2(G)},
\end{aligned}$$

that is, the operator iY is symmetric on $L^2(G)$. Hence, for all $[\xi] \in \widehat{G}$ we can choose a representative ξ such that $\sigma_{iY}(\xi)$ is a diagonal matrix, with entries $\lambda_m \in \mathbb{R}$, $1 \leq m \leq d_\xi$, which follows because symmetric matrices can be diagonalized by unitary matrices. By Proposition 1.38,

$$\sigma_X(\xi)_{mn} = i\lambda_m\delta_{mn}, \quad \lambda_j \in \mathbb{R}. \tag{1.1}$$

Notice that $\{\lambda_m\}_{m=1}^{d_\xi}$ are the eigenvalues of $\sigma_{iX}(\xi)$, so they are independent of the choice of the matrix representative, since the symbol of equivalent representations are similar matrices. Moreover, since $-(\mathcal{L}_G - X^2)$ is a positive operator and commutes with X^2, we have

$$|\lambda_m(\xi)| \leq \langle\xi\rangle,$$

for all $[\xi] \in \widehat{G}$ and $1 \leq m \leq d_\xi$.

Proposition 1.39. *Let G be a compact group, $[\xi] \in \widehat{G}$ and $\{Y_1, \cdots, Y_n\}$ be a basis for $\mathfrak{g}$. There exists $C_0 > 0$ such that*

$$\|\sigma_{\partial^\alpha}(\xi)\|_{op} \leq C_0^{|\alpha|}\langle\xi\rangle^{|\alpha|}, \quad \forall\alpha \in \mathbb{N}_0^n. \tag{1.2}$$

See [10], Proposition 3.4.

1.4 Partial Fourier Series

If $G = G_1 \times G_2$ is a product of two compact Lie groups, then we can talk about partial Fourier transform. In this section we will only state the main results we are going to use about partial Fourier transform. Full details and proofs can be found in [23]. The representation theory of the product of compact groups is well behaved, in the sense that every representation in Rep(G) can be uniquely written, up to isomorphism, as the exterior tensor product of a representation in Rep(G_1) with a representation in Rep(G_2) (see Proposition 4.14 pag 82 in [9]). Let $u \in \mathcal{D}'(G)$, $\xi \in$ Rep(G_1) and $1 \leq m, n \leq d_\xi$. The mn-component of the partial Fourier coefficient of u with respect to ξ is the linear functional defined by

$$\begin{array}{rccl} \widehat{u}(\xi,\cdot)_{mn}: & C^\infty(G_2) & \to & \mathbb{C} \\ & \psi & \to & \langle \widehat{u}(\xi,\cdot)_{mn}, \psi \rangle := \left\langle u, \overline{\xi_{nm}} \times \psi \right\rangle_G. \end{array}$$

In a similar way we can also define the components of the partial Fourier coefficient of u with respect to the second variable. It can be proved that

$$\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}} = \widehat{\widehat{u}}(\xi,\eta)_{rs_{mn}} = \widehat{u}(\xi \otimes \eta)_{ij},$$

for all $([\xi],[\eta]) \in \widehat{G}$, $1 \leq m, n \leq d_\xi$, $1 \leq r, s \leq d_\eta$, with $i = d_\eta(m-1)+r$ and $j = d_\eta(n-1)+s$.

We can also characterize smooth functions and distributions using only partial Fourier coefficients.

Proposition 1.40. *Let $\{\widehat{f}(\cdot,\eta)\}_{[\eta]\in\widehat{G}_2}$ be a sequence of functions defined on G_1. Define*

$$f(x_1,x_2) := \sum_{[\eta]\in\widehat{G}_2} d_\eta \sum_{r,s=1}^{d_\eta} \widehat{f}(x_1,\eta)\eta_{rs}(x_1).$$

Then $f \in C^\infty(G)$ if and only if $\widehat{f}(\cdot,\eta)_{rs} \in C^\infty(G_1)$ for all $[\eta] \in \widehat{G}_2$, $1 \leq r, s \leq d_\eta$, and for every multi-index β and $\ell > 0$ there exists $C_{\beta\ell} > 0$ such that

$$|\partial^\beta \widehat{f}(x_1,\eta)_{rs}| \leq C_{\beta\ell}\langle\eta\rangle^{-\ell}, \forall x_1 \in G_1, [\eta] \in \widehat{G}_2, 1 \leq r, s \leq d_\eta.$$

Proposition 1.41. *Let $\{\widehat{u}(\cdot,\eta)_{rs}\}_{[\eta]\in\widehat{G}_2}$ be a sequence of distributions on G_1. Define*

$$u := \sum_{[\eta]\in\widehat{G}_2} \sum_{r,s=1}^{d_\eta} \widehat{u}(\cdot,\eta)_{rs}\eta_{rs}.$$

Then $u \in \mathcal{D}'(G)$ if and only if there exists $K \in \mathbb{N}$ and $C > 0$ such that

$$|\langle \widehat{u}(\cdot,\eta)_{rs}, \varphi \rangle| \leq C p_K(\varphi)\langle\eta\rangle^K,$$

for all $\varphi \in C^\infty(G_1)$ and $[\eta] \in \widehat{G}_2$, where $p_K(\varphi) = \sum_{|\beta|\leq K} \|\partial^\beta\varphi\|_{L^\infty(G_1)}$.

In [22], similar results are proved on the Komatsu classes of both Roumieu and Beurling types. In our work we are only concerned about the particular case of analytic functions, which is Roumieu type with $M_k = k!$. In this case the result is the following:

Proposition 1.42. *Let $f \in C^\infty(G)$. We have that $f \in C^\omega(G)$ if and only if $\widehat{f}(\cdot,\eta)_{rs} \in C^\omega(G_1)$ for every $[\eta] \in \widehat{G}_2$, $1 \leq r, s \leq d_\eta$, and there exists $C, B > 0$ such that*

$$|\widehat{f}(x_1,\eta)_{rs}| \leq Ce^{-B\langle\eta\rangle},$$

for all $x_1 \in G_1$, $[\eta] \in \widehat{G}_2$, $1 \leq r, s \leq d_\eta$.

Remark 1.43. *When the functions $t \in \mathbb{T}^1 \mapsto \widehat{f}(\cdot,\ell)_{mn}$ are real analytic, it follows from Cauchy's integral formula that it is enough to obtain estimates for f itself and then the ones for its derivatives are consequences of this formula.*

Chapter 2

Preliminaries: Representation Theory and Fourier Analysis on $SU(2)$

In the first section we present the group $SU(2)$ and an important coordinated chart on it, whose coordinates are called Euler angles. In the second section we present two important basis of the Lie algebra $\mathfrak{su}(2)$. One of them will define the most important vector field on this work, which is called the neutral operator (recall that elements of the Lie algebra can be seen as left-invariant operators acting on functions). On third section we present the classical theory that completely describes continuous irreducible representations of $SU(2)$. This will give rise to a basis of $L^2(SU(2))$ and we show how the vectors of the basis obtained on second section acts in this basis. Finally, on the last section we summarize the important results at the end of first chapter in the cases $G = SU(2)$ and $G = \mathbb{T}^1 \times SU(2)$, which are the main Lie groups we are going to deal with. Here we follow very closely to [25].

2.1 The Lie group $SU(2)$ and the Euler angles

We recall that $SU(2) = \{u \in \mathbb{C}^{2\times 2}; u \cdot u^* = 1$ and $\det(u) = 1\}$. This is a group with the usual product of matrices and it is obviously compact. It is a closed subgroup of the general Lie group $GL(2, \mathbb{C})$. It is easy to see that every element $u \in SU(2)$ can be written in a unique way as

$$u = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$$

with $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$, that is, $(\alpha, \beta) \in \mathbb{S}^3 \subset \mathbb{C}^2 = \mathbb{R}^4$. In other words, this is an identification of $SU(2)$ with $\mathbb{S}^3$. It is clear that this identification is a diffeomorphism. It is actually more than that, it is an isomorphism of Lie groups where in $\mathbb{S}^3$ we consider the restriction of the quaternionic product of $\mathbb{R}^4$. We are not going to distinguish the matrix version of an element of $SU(2)$ with its vector version on $\mathbb{S}^3$.

Given an element $u = (\alpha, \beta) \in \mathbb{S}^3$, we can write $\alpha = r_1 e^{is}, \beta = r_2 e^{it}$ for some $r_1, r_2, t, s \in \mathbb{R}$. Since $|\alpha|^2 + |\beta|^2 = 1$, we have $r_1^2 + r_2^2 = 1$, so there exists $r \in \mathbb{R}$ such that $r_1 = \cos r, r_2 = \sin r$. In this way, $u = ((\cos r)e^{is}, (\sin r)e^{it})$. But do note that the vector $((\cos r)e^{is}, i(\sin r)e^{it})$ also lives in $\mathbb{S}^3$, therefore we can write

$$SU(2) = \left\{ \begin{bmatrix} (\cos r)e^{is} & i(\sin r)e^{it} \\ i(\sin r)e^{-it} & (\cos r)e^{-is} \end{bmatrix} ; r, s, t \in \mathbb{R} \right\}.$$

Putting $r = 0$, we get a 1-parameter subgroup of matrices with the form

$$\begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix}.$$

By putting $s = t = 0$, we get another 1-parameter subgroup of matrices with the form

$$\begin{bmatrix} \cos r & i\sin r \\ i\sin r & \cos r \end{bmatrix}.$$

Note that both of them are isomorphic to the Lie group $\mathbb{T}^1$ (they are maximal tori on $\mathbb{S}^3$). If $r, s, t \in \mathbb{R}$, then

$$\begin{bmatrix} (\cos r)e^{is} & i(\sin r)e^{it} \\ i(\sin r)e^{-it} & (\cos r)e^{-is} \end{bmatrix} = \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix} \cdot \begin{bmatrix} (\cos r) & i(\sin r) \\ i(\sin r) & (\cos r) \end{bmatrix} \cdot \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}.$$

So every element in $SU(2)$ can be decomposed as the product above.

Definition 2.1. *The Euler angles (ϕ, θ, ψ) with $0 \leq \phi < 2\pi$, $0 \leq \theta < \pi$, $-2\pi \leq \psi < 2\pi$ are the real numbers that corresponds to the matrix*

$$u(\phi, \theta, \psi) = \begin{bmatrix} \cos(\frac{\theta}{2})e^{i(\frac{\phi+\psi}{2})} & i\sin(\frac{\theta}{2})e^{i(\frac{\phi-\psi}{2})} \\ i\sin(\frac{\theta}{2})e^{-i(\frac{\phi-\psi}{2})} & \cos(\frac{\theta}{2})e^{-i(\frac{\phi+\psi}{2})} \end{bmatrix}$$

These intervals are sufficient to cover all the elements in $SU(2)$ and it is injective almost everywhere. If $u = (\alpha, \beta) \in \mathbb{S}^3$, we can recover the angles using the relations

$$\begin{cases} 2\alpha\bar{\alpha} &= 1 + \cos\theta \\ 2\alpha\beta &= ie^{i\phi}\sin\theta \\ -2\alpha\bar{\beta} &= ie^{i\psi}\sin\theta \end{cases}$$

Now let us explore a little the relation between $SU(2)$ and the quaternions. The set of quaternions $\mathbb{H}$ is just $\mathbb{R}^4$ with a product operation analogous to the standard one of $\mathbb{C}$. We write vectors $u = (a, b, c, d)$ on $\mathbb{R}^4$ on the form $u = a + bi + cj + dk$ with the symbols i, j, k satisfying $i^2 = j^2 = k^2 = -1, ij = k, ki = j, jk = i$ and make products between vectors by declaring it as a distributive product. This makes $(\mathbb{H}, +, \cdot)$ into an associative and non-commutative algebra. If we define the conjugated quaternion by $\bar{u} = a - bi - cj - dk$, then $u \cdot \bar{u} = \|u\|^2 = (a^2 + b^2 + c^2 + d^2)^2$. In particular, if $u \neq 0$, then u is invertible with respect to the quaternionic product and $u^{-1} = \frac{\bar{u}}{\|u\|^2}$. Also, this product satisfies $\|u \cdot v\| = \|u\| \cdot \|v\|$ for all $u, v \in \mathbb{H}$. This implies that $\mathbb{S}^3 = \{u \in \mathbb{H}; \|u\| = 1\}$ is a group with the quaternionic product. The identification between $(\mathbb{S}^3, \cdot)$ and $SU(2)$ is given by

$$a + bi + cj + dk \mapsto \begin{bmatrix} a + id & b + ic \\ -b + ic & a - id \end{bmatrix}$$

Via de Euler angles, this identification is

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \cos(\frac{\theta}{2}) \cos(\frac{\phi+\psi}{2}) \\ -\sin(\frac{\theta}{2}) \sin(\frac{\phi-\psi}{2}) \\ \sin(\frac{\theta}{2}) \cos(\frac{\phi-\psi}{2}) \\ \cos(\frac{\theta}{2}) \sin(\frac{\phi+\psi}{2}) \end{bmatrix},$$

with the parameters $0 \leq \phi < 2\pi, 0 \leq \theta < \pi, -2\pi \leq \psi < 2\pi$.

The integral of measurable functions on $SU(2)$ with respect to the Haar measure can also be written in terms of Euler angles ([25] pg 605). It can be shown that if $f \in L^1(SU(2))$, then

$$\int_{SU(2)} f(x)dx = \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{\pi} \int_{-\pi}^{\pi} f(u(\phi, \theta, \psi)) \sin(\theta) d\phi d\theta d\psi.$$

2.2 Invariant differential operators on $SU(2)$

Now we are going to explicit some interesting basis of the Lie algebra $\mathfrak{su}(2)$ and state the main results about the behavior of the corresponding invariant vector fields acting on $SU(2)$. Their expressions in Euler-angles coordinates are also presented.

Consider the 1-parameter subgroups $\omega_1, \omega_2, \omega_3 : \mathbb{R} \to SU(2)$ given by

$$\omega_1(t) = \begin{bmatrix} \cos(t/2) & i\sin(t/2) \\ i\sin(t/2) & \cos(t/2) \end{bmatrix}, \omega_2(t) = \begin{bmatrix} \cos(t/2) & -\sin(t/2) \\ \sin(t/2) & \cos(t/2) \end{bmatrix}, \omega_3(t) = \begin{bmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{bmatrix},$$

and also define $w_j = \omega_j(\pi/2)$, so that

$$w_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, w_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, w_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

The Lie algebra vectors $Y_j := \omega_j'(0)$ are given by

$$Y_1 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, Y_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Y_3 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

It is easy to see that the set with these matrices form a basis of $\mathfrak{su}(2)$. Their brackets are given by $[Y_1, Y_2] = Y_3, [Y_2, Y_3] = Y_1$ and $[Y_3, Y_1] = Y_2$. The three 1-parameter subgroups ω_j are all conjugated to each other since

$$\omega_3(t) = w_1\omega_2(t)w_1^{-1}, \quad \omega_1(t) = w_2\omega_3(t)w_2^{-1}, \quad \omega_2(t) = w_3\omega_1(t)w_3^{-1}.$$

In particular, taking the derivative at $t = 0$ in each of the above relations we get

$$Y_3 = w_1Y_2w_1^{-1}, \quad Y_1 = w_2Y_3w_2^{-1}, \quad Y_2 = w_3Y_1w_3^{-1}.$$

Given $Y \in \mathfrak{su}(2)$, we denote by $D_Y : C^\infty(SU(2)) \to C^\infty(SU(2))$ the left-invariant vector field corresponding to Y. We will write $D_j := D_{Y_j}, j = 1, 2, 3$.

Proposition 2.2. *In terms of Euler-angles we have*

$$\begin{aligned} D_1 &= +\cos\psi\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\cos\theta}{\sin\theta}\sin\psi\frac{\partial}{\partial\psi}; \\ D_2 &= -\sin\psi\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\cos\theta}{\sin\theta}\cos\psi\frac{\partial}{\partial\psi}; \\ D_3 &= \frac{\partial}{\partial\psi} \end{aligned}$$

with $0 < \theta < \pi$.

Just to give a sketch of the proof, if $f \in C^\infty(SU(2))$ and $u \in SU(2)$, then

$$D_j(f)(u) = \left.\frac{d}{dt}\right|_{t=0} f(u\cdot\exp(tY_j)) = \left.\frac{d}{dt}\right|_{t=0} f(u\cdot\omega_j(t)),$$

which in Euler-angles coordinates is

$$\left.\frac{d}{dt}\right|_{t=0} f(\tilde{u}) = \left(\phi'(0)\frac{\partial}{\partial\phi} + \theta'(0)\frac{\partial}{\partial\theta} + \psi'(0)\frac{\partial}{\partial}\right) f(\tilde{u}(\phi,\theta,\psi)\Big|_{t=0}$$

where $\tilde{u}(\phi(t), \theta(t), \psi(t)) = u(\phi_0, \theta_0, \psi_0)\cdot\omega_j(t)$. So one must use a formula for the Euler-angles coordinates of a product of two elements in $SU(2)$ and then take the derivative of that expression at $t = 0$. All the details can be found in [25] pg 601.

The basis $\{Y_1, Y_2, Y_3\}$ of $\mathfrak{su}(2)$ is a good basis for the Laplacian, in the sense that in this basis $\mathcal{L}$ is a sum of squares, as we will see later. But, there is another interesting basis of $\mathfrak{su}(2)$ that we are going to use. Consider the operators

$$\partial_+ = iD_1 - D_2, \quad \partial_- = iD_1 + D_2, \quad \partial_0 = iD_3.$$

In the literature, they are called creation, annihilation and neutral operators, respectively. When dealing with root systems of the Lie algebra $\mathfrak{su}(2)$, each of these three elements are generators of the center of the Lie algebra, the positive roots and negative roots, respectively. The inverse relations are

$$D_1 = -\frac{i}{2}(\partial_- + \partial_+), \quad D_2 = \frac{1}{2}(\partial_- - \partial_+), \quad D_3 = -i\partial_0.$$

We also have $\mathcal{L} = -\partial_0^2 - \frac{1}{2}(\partial_+\partial_- + \partial_-\partial_+)$. Their brackets are given by

$$[\partial_0, \partial_+] = \partial_+, [\partial_-, \partial_0] = \partial_-, [\partial_+, \partial_-] = 2\partial_0.$$

Finally, with respect to Euler-angles coordinates we have

$$\begin{aligned}
\partial_+ &= e^{-i\psi}\left(i\frac{\partial}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\psi}\right)\\
\partial_- &= e^{i\psi}\left(i\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\psi}\right)\\
\partial_0 &= i\frac{\partial}{\partial\psi}.
\end{aligned}$$

2.3 Irreducible representations of $SU(2)$

A natural space to look for representations of a matrix group is a space of polynomials, with the number of variables equal to the order of the matrices. We identify $z = (z_1, z_2) \in \mathbb{C}^2$ with the matrix $(z_1\ \ z_2) \in \mathbb{C}^{1\times 2}$ and consider the map $T : SU(2) \to GL(\mathbb{C}[z_1, z_2])$ defined by

$$(Tu)(f)(z) = f(z \cdot u).$$

In this way, $\mathbb{C}[z_1, z_2]$ is a representation of $SU(2)$. Also note that f and $(Tu)(f)$ have the same degree. In particular, Tu preserves the subspace V_ℓ of homogeneous polynomials of degree 2ℓ, for each $\ell \in \frac{1}{2}\mathbb{N}_0$. We denote by $T_\ell : SU(2) \to GL(V_\ell)$ the restriction of T to the T-invariant subspace V_ℓ. It is clear that dim $V_\ell = 2\ell + 1$ and $\{p_{\ell k}; k = 0, 1, ..., 2\ell\}$, with $p_{\ell k}(z) = z_1^k z_2^{2\ell-k}$, is a natural basis for V_ℓ.

Proposition 2.3. *The representation T_ℓ is irreducible. Besides, each $p_{\ell k}$ is an eigenvector of $T_\ell(u(\phi, 0, 0))$, with corresponding eigenvalue $e^{i\phi(k-\ell)}$.*

Proof. The idea is to use the converse of the Schur lemma, which is true in this case because $SU(2)$ is compact. Let $A \in \text{End}(V_\ell)$ such that $T_\ell(u) \circ A = A \circ T_\ell(u)$ for all $u \in SU(2)$. We want to show that A is a scalar multiple of identity map of V_ℓ. Let

$$u = u(2s, 0, 0) = \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix},$$

then

$$\begin{aligned} T_\ell(u(2s,0,0))(p_{\ell k})(z_1, z_2) &= p_{\ell k}((z_1 \;\; z_2) \cdot u(2s,0,0)) \\ &= p_{\ell k}(e^{is}z_1, e^{-is}z_2) \\ &= (e^{is}z_1)^k(e^{-is}z_2)^{2\ell-k} \\ &= e^{is2(k-\ell)}p_{\ell k}. \end{aligned}$$

That is, $T_\ell(u(2s,0,0))(p_{\ell k}) = e^{2(k-\ell)is}$. By the hypothesis, we have

$$\begin{aligned} T_\ell(u(2s,0,0))A(p_{\ell k}) &= A(T_\ell(u(2s,0,0)).p_{\ell k}) \\ &= A(e^{2(k-\ell)is}p_{\ell k}) \\ &= e^{2(k-\ell)is}A(p_{\ell k}), \end{aligned}$$

so $A(p_{\ell k})$ is an eigenvector of $T_\ell(u(2s,0,0))$ with eigenvalue $e^{2(k-\ell)is}$. If $A(p_{\ell k}) = \sum_j a_j p_{\ell j}$, then

$$\begin{aligned} e^{2(k-\ell)is}A(p_{\ell k}) &= T_\ell(u(2s,0,0))(A(p_{\ell k})) \\ &= T_\ell(u(2s,0,0))\textstyle\sum_j a_j p_{\ell k} \\ &= \textstyle\sum_j a_j e^{2(j-\ell)is}p_{\ell j}. \end{aligned}$$

On the other hand, $e^{2(k-\ell)is}A(p_{\ell k}) = e^{2(k-\ell)is}\sum_j a_j p_{\ell k} = \sum_j (e^{2(k-\ell)is}a_j)p_{\ell k}$. Since $\{p_{\ell k}\}$ is a basis, we have $e^{2(j-\ell)is}a_j = e^{2(k-\ell)is}a_j$ for all $j = 0, 1, ..., 2\ell$. If $a_j \neq 0$ for some $j \neq k$, then $e^{2(j-\ell)is} = e^{2(k-\ell)is}$, which implies there exist $n \in \mathbb{Z}$ such that $2s(j-k) - 2s(k-\ell) = 2n\pi$, that is, $s(j-k) = n\pi$. We can choose s small enough such that this equality implies $j = k$, so $A(p_{\ell k}) = a_k p_{\ell k}$. Now we are going to prove that all a_k are actually equal for all k. If now

$$u = u(0, 2r, 0) = \begin{bmatrix} \cos r & i\sin r \\ i\sin r & \cos r \end{bmatrix},$$

then

$$\begin{aligned}
T_\ell(u(0,2r,0)(A_{p\ell 0}(z)) &= T_\ell(u(0,2r,0))(a_0 p_{\ell 0}(z)) \\
&= a_0 p_{\ell 0}(z\cdot u(0,2r,0)) \\
&= a_0(iz_1\sin r + z_2\cos r)^{2\ell} \\
&= a_0\sum_k \binom{2\ell}{k}(iz_1\sin r)^k(z_2\cos r)^{2\ell-k} \\
&= a_0\sum_k \binom{2\ell}{k} i^k(\sin r)^k(\cos r)^{2\ell-k}p_{\ell k}(z) \\
&= \sum_k \binom{2\ell}{k} i^k(\sin r)^k(\cos r)^{2\ell-k}a_0 p_{\ell k}(z).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A(T_\ell(u(0,2r)))(p_{\ell 0})(z) &= A\left(\sum_k \binom{2\ell}{k} i^k(\sin r)^k(\cos r)^{2\ell-k}p_{\ell k}(z)\right) \\
&= \sum_k \binom{2\ell}{k} i^k(\sin r)^k(\cos r)^{2\ell-k}a_k p_{\ell k}(z).
\end{aligned}$$

So, if we choose r such that $\sin r \neq 0 \neq \cos r$, then we conclude that $a_k = a_0$ for all k. In this way, $A(p_{\ell k}) = a_0 p_{\ell k}$ for all k, and then $A = a_0\cdot \mathrm{Id}_{V_\ell}$. $\square$

Lemma 2.4. *For each $g = (\alpha,\beta) \in SU(2)$, there exists a decomposition $g = uhu^{-1}$ such that $u, h \in SU(2)$ and h is diagonal.*

Proof. If $\beta = 0$, the result is trivial, with $h = g$ and $u = \mathrm{Id}$, so let us suppose that $\beta \neq 0$. The characteristic polynomial of u is $P_g(z) = z^2 - 2\mathrm{Re}(\alpha).z + 1$, with $\Delta = 4(\mathrm{Re}(\alpha)^2 - 1)$. Since $|\beta| \neq 0$, from $|\alpha|^2 + |\beta|^2 = 1$ we get $|\alpha|^2 < 1$, which implies $|\mathrm{Re}(\alpha)|^2 < 1$. So P_g two distinct roots, let us say $z_1, z_2 \in \mathbb{C}$. Define

$$h = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}.$$

Since $1 = \det(g) = z_1\cdot z_2 = \det(h)$, we have $h \in SU(2)$. Let $v \in \mathbb{C}^{2\times 2}$ be a matrix such that its collums are eigenvectors corresponding to z_1 and z_2 respectively, so $g = vhv^{-1}$. Since $z_1 \neq z_2$, we have $v \in U(2)$, so rescaling v we can suppose that $\det(v) = 1$ and the relation $g = vhv^{-1}$ still holds. $\square$

Theorem 2.5. *Let $T_\infty : SU(2) \to GL(V)$ be a finite dimensional irreducible representation of $SU(2)$ and let $\ell \in \frac{1}{2}\mathbb{N}_0$ such that $\dim(V) = 2\ell + 1$. Then, T_∞ is equivalent to T_ℓ.*

Proof. Let $m \in \frac{1}{2}\mathbb{N}_0 \cup \{\infty\}$ and $\chi_m : SU(2) \to \mathbb{C}$ be the character of the representation T_m. From the orthogonality of the characters of irreducible representations, it is enough to prove that there exists $\ell \in \frac{1}{2}\mathbb{N}_0$ such that $\langle \chi_\infty, \chi_\ell \rangle_{L^2} \neq 0$. Since χ_m is a trace, we have that $\chi_m(uhu^{-1}) = \chi_m(h)$ for all $u, h \in SU(2)$. By the previous lemma, every element in $SU(2)$ can be written as a product of this form with elements still living on $SU(2)$, so we can identify χ_m with the function $f_m : \mathbb{R} \to \mathbb{C}$ defined by $f_m(t) = \chi_m(h(t))$, where $h(t) = (e^{it}, e^{-it}) \in SU(2)$. Basically the more general fact that is behind this argument is that the restriction of a character of a finite dimensional irreducible representation to a maximal tori completely characterize the representation. On the one hand,

$$\langle \chi_\infty, \chi_\ell \rangle_{L^2} = \int_{SU(2)} \chi_\infty(g)\overline{\chi_\ell(g)} d\mu_{SU(2)}(g) = \frac{1}{2\pi} \int_0^{2\pi} f_\infty(t)\overline{f_\ell(t)} dt.$$

On the other hand,

$$T_\ell(h(t))(p_{\ell k})(z) = p_{\ell k}(z \cdot h(t)) = p_{\ell k}(z_1 e^{it}, z_2 e^{-it}) = e^{2it(k-\ell)} p_{\ell k}(z),$$

which implies $f_\ell(t) = \chi_\ell(u) = \mathrm{Tr}(T_\ell(h(t))) = \sum_{k=-\ell}^{\ell} e^{itk}$, so the set $\{f_\ell\}_{\ell \in \frac{1}{2}\mathbb{N}_0}$ is dense in $L^2[0, 2\pi]$. Since $\chi_\infty(\mathrm{Id}) = \dim(V) \neq 0$, there exists $\ell \in \frac{1}{2}\mathbb{N}_0$ such that $\langle \chi_\infty, \chi_\ell \rangle \neq 0$. $\square$

Given $p, q \in V_\ell$, we define $\langle p, q \rangle = \left\langle p|_{SU(2)}, q|_{SU(2)} \right\rangle_{L^2(SU(2))}$. It is easy to see that T_ℓ becomes an unitary representation of $SU(2)$ with this inner product. However, the basis $\{p_{\ell k}\}$ is not orthogonal with respect to this inner product. Instead, we consider the set

$$\{q_{\ell k}; k = -\ell, -\ell + 1, ..., \ell - 1, \ell\},$$

with

$$q_{\ell k}(z_1, z_2) = \frac{z_1^{\ell-k} z_2^{\ell+k}}{\sqrt{(\ell - k)!(\ell + k)!}}.$$

Using Euler-angles coordinates, it can be seem that this set is in fact an orthonormal basis of V_ℓ. Now or goal is to present the matrix of the linear operator $T_\ell(u)$ in this basis, for each $u \in SU(2)$. We will denote by $\mathsf{t}^\ell_{mn}(u)$ the entries of this matrix, with $-\ell \leq m, n \leq \ell$ being half integers. If $u = u(\phi, \theta, \psi)$, we will write $\mathsf{t}^\ell_{mn}(u) = \mathsf{t}^\ell_{mn}(\phi, \theta, \psi)$. Right after the following Theorem, we will see how the vector fields D_j act on these functions.

Theorem 2.6. *If $u = u(\phi, \theta, \psi)$, then*

$$\mathsf{t}^\ell_{mn}(u) = \left(\frac{d}{dz_1}\right)^{\ell-m} \left(\frac{d}{dz_2}\right)^{\ell+m} \frac{(z_1 a + z_2 c)^{\ell-n}(z_1 b + z_2 d)^{\ell+n}}{\sqrt{(\ell - m)!(\ell + m)!(\ell - n)!(\ell + n)!}},$$

where the right hand side does not actually depend on (z_1, z_2). Moreover:

$$\mathsf{t}^\ell_{mn}(\phi, \theta, \psi) = e^{-i(m\phi+n\psi)} P^\ell_{mn}(\cos\theta),$$

where

$$P^\ell_{mn}(x) = c^\ell_{mn} \frac{(1-x)^{(n-m)/2}}{(1+x)^{(n+m)/2}} \left(\frac{d}{dx}\right)^{\ell-m} \left[(1-x)^{\ell-n}(1+x)^{\ell+n}\right]$$

with

$$c^\ell_{mn} = 2^{-\ell} \frac{(-1)^{\ell-n} i^{n-m}}{\sqrt{(\ell-n)!(\ell+n)!}} \sqrt{\frac{(\ell+m)!}{(\ell-m)!}}.$$

Proof. For each $-\ell \le n \le \ell$ and $u \in SU(2)$ we have

$$T_\ell(u)(q_{\ell n}) = \sum_{m=-\ell}^{\ell} \mathsf{t}^\ell_{mn}(u) q_{\ell m}.$$

So for all $z = (z_1, z_2) \in \mathbb{C}^2$, one on hand we have

$$\begin{aligned} T_\ell(u)(q_{\ell n})(z) &= \sum_m \mathsf{t}^\ell_{mn}(u) q_{\ell m}(z) \\ &= \sum_m \mathsf{t}^\ell_{mn}(u) \frac{z_1^{\ell-m} z_2^{\ell+m}}{\sqrt{(\ell-m)!(\ell+m)!}}, \end{aligned}$$

and on the another hand,

$$T_\ell(u)(q_{\ell n})(z) = q_{\ell n}(z \cdot u) = \frac{(z_1 a + z_2 c)^{\ell-n}(z_1 b + z_2 d)^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}}.$$

Now if we choose an index m, after taking sucessive derivatives on both sides with respect to z_1 and z_2 until we get a degree 0 polynomial we get

$$\begin{aligned} \left(\tfrac{d}{dz_1}\right)^{\ell-m} \left(\tfrac{d}{dz_2}\right)^{\ell+m} (q_{\ell n}(z \cdot u)) &= \sum_m \mathsf{t}^\ell_{mn}(u) \left(\frac{d}{dz_1}\right)^{\ell-m} \left(\frac{d}{dz_2}\right)^{\ell+m} \frac{z_1^{\ell-m} z_2^{\ell+m}}{\sqrt{(\ell-m)!(\ell+m)!}} \\ &= \mathsf{t}^\ell_{mn}(u) \tfrac{(\ell-m)!(\ell+m)!}{\sqrt{(\ell-m)!((\ell+m)!)}} \\ &= \mathsf{t}^\ell_{mn}(u) \sqrt{(\ell-m)!(\ell+m)!)}, \end{aligned}$$

so

$$\mathsf{t}^\ell_{mn}(u) = \left(\frac{d}{dz_1}\right)^{\ell-m} \left(\frac{d}{dz_2}\right)^{\ell+m} \frac{(z_1 a + z_2 c)^{\ell-n}(z_1 b + z_2 d)^{\ell+n}}{\sqrt{(\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!}}.$$

Since there is a total of 2ℓ derivatives on the right hand side and the polynomial is homogeneous of degree 2ℓ, the right hand side is a constant (exactly the coefficient of the monomial $z_1^{\ell-m} z_2^{\ell+m}$). Now for the second part we use the Newton's binomial formula to write

$$\left(\frac{d}{dz_1}\right)^{\ell-m} \left(\frac{d}{dz_2}\right)^{\ell+m} \left[(z_1 a + z_2 c)^{\ell-n}(z_1 b + z_2 d)^{\ell+n}\right] =$$

$$= \left(\frac{d}{dz_1}\right)^{\ell-m}\left(\frac{d}{dz_2}\right)^{\ell+m}\sum_{i=0}^{\ell-n}\sum_{j=0}^{\ell+n}\binom{\ell-n}{i}\binom{\ell+n}{j}a^i b^j c^{\ell-n-i}d^{\ell+n-j}z_1^{i+j}z_2^{2\ell-i-j}.$$

Take all the derivatives on the variable z_2 and then calculate it at $z_2 = 1$. Rename the variable z_1 to y and calculate the derivative at $y = 0$ to get:

$$\left.\frac{d}{dy}^{\ell-m}\right|_{y=0}(\ell+m)!\sum_{i=0}^{\ell-n}\sum_{j=0}^{\ell+n}a^i b^j c^{\ell-n-i}d^{\ell+n-j}y^{i+j} = (\ell+m)!\left.\frac{d}{dy}^{\ell-m}\right|_{y=0}\left[(ay+c)^{\ell-n}(by+d)^{\ell+n}\right].$$

Since $u \in SU(2)$, $ad - bc = 1$, we have $a(by + d) - b(ay + c) = aby + ad - bay - bc = 1$. This inspire the change of variables $a(by + d) = (x + 1)/2$. Therefore

$$b(ay + c) = a(by + d) - 1 = (x + 1)/2 - 1 = (x - 1)/2,$$

and so $x = 1 + 2aby + 2bc$, which implies $\frac{dx}{dy} = 2ab$. Hence, we get

$$\begin{aligned}
\mathsf{t}^{\ell}_{mn}(u) &= \frac{(\ell+m)!}{\sqrt{(\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!}}\left.\frac{d}{dy}^{\ell-m}\right|_{y=0}\left[(ay+c)^{\ell-n}(by+d)^{\ell+n}\right]\\
&= \frac{(\ell+m)!(2ab)^{\ell-m}}{\sqrt{(\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!}}\left.\frac{d}{dx}^{\ell-m}\right|_{x=2ad-1}\left[\left(\frac{x-1}{2b}\right)^{\ell-n}\left(\frac{x+1}{2a}\right)^{\ell+n}\right]\\
&= \frac{2^{-\ell-m}(\ell+m)!}{\sqrt{(\ell-m)!(\ell+m)!(\ell-n)!(\ell+n)!}}\frac{b^{n-m}}{a^{m+n}}\left.\frac{d}{dx}^{\ell-m}\right|_{x=2ad-1}\left[(x-1)^{\ell-n}(x+1)^{\ell+n}\right].
\end{aligned}$$

Now using Euler–angles coordinates we have

$$x = 2ad - 1 = 2e^{i(\phi+\psi)/2}\cos\left(\tfrac{\theta}{2}\right)e^{-i(\phi+\psi)/2}\cos\left(\tfrac{\theta}{2}\right) - 1 = 2\cos^2\left(\tfrac{\theta}{2}\right) - 1 = \cos\theta.$$

Moreover, $\cos^2(\theta/2) = (x + 1)/2$ implies that

$$\sin^2(\theta/2) = 1 - \cos^2(\theta/2) = 1 - (x + 1)/2 = (1 - x)/2.$$

So,

$$\begin{aligned}
\mathsf{t}^{\ell}_{mn}(u) &= \\
&= \sqrt{\tfrac{(\ell+m)!}{(\ell-m)!(\ell-n)!(\ell+n)!}}\frac{\left(\frac{1-x}{2}\right)^{(n-m)/2}}{\left(\frac{1+x}{2}\right)^{(m+n)/2}}2^{-\ell-m}i^{n-m}e^{-i(n\phi+m\psi)}\left.\tfrac{d}{dx}^{\ell-m}\right|_{x=\cos\theta}[\cdots]\\
&= \sqrt{\tfrac{(\ell+m)!}{(\ell-m)!(\ell-n)!(\ell+n)!}}2^{-\ell}i^{n-m}\tfrac{(1-x)^{(n-m)/2}}{(1+x)^{(n+m)/2}}\left.\frac{d}{dx}^{\ell-m}\right|_{x=\cos\theta}\left[(x-1)^{\ell-n}(x+1)^{\ell+n}\right]e^{-i(n\phi+m\psi)}\\
&= P^{\ell}_{mn}(\cos\theta)e^{-i(n\phi+m\psi)}.
\end{aligned}$$

$\square$

Proposition 2.7. *The way the vector fields D_1, D_2, D_3 act on the functions t^{ℓ}_{mn} is the following:*

$$\begin{aligned}
D_1\mathsf{t}^{\ell}_{mn} &= \tfrac{\sqrt{(\ell-n)(\ell+n+1)}}{-2i}\mathsf{t}^{\ell}_{m,n+1} + \tfrac{\sqrt{(\ell+n)(\ell-n+1)}}{-2i}\mathsf{t}^{\ell}_{m,n-1}\\
D_2\mathsf{t}^{\ell}_{mn} &= \tfrac{\sqrt{(\ell-n)(\ell+n+1)}}{2}\mathsf{t}^{\ell}_{m,n+1} - \tfrac{\sqrt{(\ell+n)(\ell-n+1)}}{2}\mathsf{t}^{\ell}_{m,n-1}\\
D_3\mathsf{t}^{\ell}_{mn} &= -in\mathsf{t}^{\ell}_{mn}.
\end{aligned}$$

Proof. Recall that $(D_j f)(u) = (d/dt)|_{t=0} f(u \cdot \omega_j(t))$ and $T_\ell(u)(q_{\ell n}) = \sum_m \mathsf{t}^\ell_{mn}(u) q_{\ell m}$. For each fixed $z \in \mathbb{C}^2$, we can act the operator D_j on the expression $q_{\ell n}(z \cdot u) = \sum_m \mathsf{t}^\ell_{mn}(u) q_{\ell m}(z)$ with respect to the variable u, obtaining:

$$D_j(u \mapsto q_{\ell n}(z \cdot u)) = \sum_m D_j(\mathsf{t}^\ell_{mn}(u)) q_{\ell m}(z).$$

Let us start with $j = 3$:

$$\begin{aligned}
\sum_m (D_3 \mathsf{t}^\ell_{mn})(u) q_{\ell m}(z) &= \frac{d}{dt}\Big|_{t=0} q_{\ell n}(z \cdot (u\omega_3(t))) \\
&= \frac{d}{dt}\Big|_{t=0} \frac{\left((zu)_1 e^{it/2}\right)^{\ell-n} \left((zu)_2 e^{-it/2}\right)^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}} \\
&= \frac{d}{dt}\Big|_{t=0} e^{-itn} q_{\ell n}(zu) \\
&= -in q_{\ell n}(zu) \\
&= -in T_\ell(u)(q_{\ell n})(z) \\
&= \sum_m (-in) \mathsf{t}^\ell_{mn}(u) q_{\ell m}(z)
\end{aligned}$$

hence, comparing coefficients we have

$$D_3\left(\mathsf{t}^\ell_{mn}\right) = -in \mathsf{t}^\ell_{mn}.$$

Now for $j = 2$:

$$\begin{aligned}
D_2(q_{\ell n}(zu)) &= \frac{d}{dt}\Big|_{t=0} q_{\ell n}(z(u\omega_2(t))) \\
&= \frac{d}{dt}\Big|_{t=0} \frac{\left[(zu)_1 \cos(\frac{t}{2}) + (zu)_2 \sin(\frac{t}{2})\right]^{\ell-n} \cdot \left[-(zu)_1 \sin(\frac{t}{2}) + (zu)_2 \cos(\frac{t}{2})\right]^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}} \\
&= \frac{(\ell-n)(zu)_1^{\ell-n-1} \frac{1}{2}(zu)_2 (zu)_2^{\ell+n} + (zu)_1^{\ell-n}(\ell+n)(zu)_2^{\ell+n-1} \cdot (-\frac{1}{2})(zu)_1}{\sqrt{(\ell-n)!(\ell+n)!}} \\
&= \frac{1}{2} \frac{(\ell-n)}{\sqrt{(\ell-n)!(\ell+n)!}} (zu)_1^{\ell-(n+1)} (zu)_2^{\ell+(n+1)} - \frac{1}{2} \frac{(\ell+n)}{\sqrt{(\ell-n)!(\ell+n)!}} (zu)_1^{\ell-(n-1)} (zu)_2^{\ell+(n-1)} \\
&= \frac{1}{2}\sqrt{(\ell-n)(\ell+n+1)}\, q_{\ell,n+1}(zu) - \frac{1}{2}\sqrt{(\ell+n)(\ell-n+1)}\, q_{\ell,n-1}(zu) \\
&= \sum_m \left[\tfrac{1}{2}\sqrt{(\ell-n)(\ell+n+1)}\, \mathsf{t}^\ell_{m,n+1}(u) - \tfrac{1}{2}\sqrt{(\ell+n)(\ell-n+1)}\, \mathsf{t}^\ell_{m,n-1}(u)\right] q_{\ell m}(z),
\end{aligned}$$

hence

$$D_2(\mathsf{t}^\ell_{mn})(u) = \frac{1}{2}\sqrt{(\ell-n)(\ell+n+1)}\, \mathsf{t}^\ell_{m,n+1}(u) - \frac{1}{2}\sqrt{(\ell+n)(\ell-n+1)}\, \mathsf{t}^\ell_{m,n-1}(u).$$

The calculations for D_1 are totally analogous. □

Now it is easy to see how the operators ∂_+, ∂_- and ∂_0 act on t^ℓ_{mn}. Using the above formulas we get

$$\begin{aligned}
\partial_+(\mathsf{t}^\ell_{mn}) &= (iD_1 - D_2)(\mathsf{t}^\ell_{mn}) \\
&= i\left[\frac{\sqrt{(\ell-n)(\ell+n+1)}}{-2i}\mathsf{t}^\ell_{m,n+1} + \frac{\sqrt{(\ell+n)(\ell-n+1)}}{-2i}\mathsf{t}^\ell_{m,n-1}\right] - \\
&- \left[\frac{\sqrt{(\ell-n)(\ell+n+1)}}{2}\mathsf{t}^\ell_{m,n+1} - \frac{\sqrt{(\ell+n)(\ell-n+1)}}{2}\mathsf{t}^\ell_{m,n-1}\right] \\
&= -\sqrt{(\ell-n)(\ell+n+1)}\mathsf{t}^\ell_{m,n+1}.
\end{aligned}$$

In particular, ∂_+ is upper triangular in this basis. In a very similar way we get for $\partial_- = iD_1 + D_2$ the expression

$$\partial_-(\mathsf{t}^\ell_{mn}) = -\sqrt{(\ell+n)(\ell-n+1)}\mathsf{t}^\ell_{m,n-1}.$$

So ∂_- is lower triangular in this bases. Finally, for $\partial_0 = iD_3$ we get

$$\partial_0(\mathsf{t}^\ell_{mn}) = n\, \mathsf{t}^\ell_{mn}.$$

and ∂_0 is diagonal in this basis. In another words, $\{\mathsf{t}^\ell_{mn}\}_{\ell\in\frac{1}{2}\mathbb{N}_0}$ is a basis of eigenvectors for ∂_0.

To finish this chapter, let us see how the Laplacian acts on these functions. Recall that $\mathfrak{su}(2)$ is a semisimple Lie algebra with Killing form $B(X,Y) = 4\text{tr}(XY)$. The matrix of B in the basis $\{Y_1, Y_2, Y_3\}$ is the 3×3 diagonal matrix where the diagonal entries are all equal to -2, so its inverse is also diagonal with diagonal entries equal to $-1/2$. By the definition of the Laplacian in terms of the Casimir element we have

$$\mathcal{L} = Y_1Y^1 + Y_2Y^2 + Y_3Y^3 = -\frac{1}{2}\sum_{j=1}^{3} Y_j^2.$$

Using the inverse relations $D_1 = -\frac{i}{2}(\partial_- + \partial_+)$, $D_2 = \frac{1}{2}(\partial_- - \partial_+)$ and $D_3 = -i\partial_0$ we get

$$\begin{aligned}
D_1^2 &= -\tfrac{1}{4}(\partial_- + \partial_+)\circ(\partial_- + \partial_+) = -\tfrac{1}{4}\left[\partial_-^2 + \partial_-\partial_+ + \partial_+\partial_- + \partial_+^2\right] \\
D_2^2 &= +\tfrac{1}{4}(\partial_- - \partial_+)\circ(\partial_- - \partial_+) = +\tfrac{1}{4}\left[\partial_-^2 - \partial_-\partial_+ - \partial_+\partial_- + \partial_+^2\right] \\
D_3^2 &= -\partial_0^2
\end{aligned}$$

so

$$\mathcal{L} = -\frac{1}{2}\left[-\frac{1}{2}\partial_-\partial_+ - \frac{1}{2}\partial_+\partial_- - \partial_0^2\right]$$

Calculating how each part of the expression above acts on t^ℓ_{mn}:

$$\begin{aligned}
\partial_-\partial_+\mathsf{t}^\ell_{mn} &= \partial_-\left(-\sqrt{(\ell-n)(\ell+n+1)}\mathsf{t}^\ell_{m,n+1}\right) \\
&= -\sqrt{(\ell-n)(\ell+n+1)}\left(-\sqrt{(\ell+n+1)(\ell-n-1+1)}\right)\mathsf{t}^\ell_{mn} \\
&= (\ell-n)(\ell+n+1)\mathsf{t}^\ell_{mn},
\end{aligned}$$

$$\begin{aligned}
\partial_+\partial_-\mathsf{t}^\ell_{mn} &= \partial_+\left(-\sqrt{(\ell+n)(\ell-n+1)}\mathsf{t}^\ell_{m,n-1}\right)\\
&= -\sqrt{(\ell+n)(\ell-n+1)}\left(-\sqrt{(\ell-n+1)(\ell-n-1+1)}\right)\mathsf{t}^\ell_{mn}\\
&= (\ell+n)(\ell-n+1)\mathsf{t}^\ell_{mn},
\end{aligned}$$

and

$$\partial_0^2\mathsf{t}^\ell_{mn} = \partial_0(n\mathsf{t}^\ell_{mn}) = n^2\,\mathsf{t}^\ell_{mn}.$$

Putting all together, we get

$$\begin{aligned}
\mathcal{L}(\mathsf{t}^\ell_{mn}) &= -\tfrac{1}{2}\left[-\tfrac{1}{2}\left((\ell-n)(\ell+n+1)+(\ell+n)(\ell-n+1)\right)-n^2\right]\mathsf{t}^\ell_{mn}\\
&= -\tfrac{1}{2}\left[-(\ell^2-n^2+\ell)-n^2\right]\mathsf{t}^\ell_{mn}\\
&= \frac{\ell(\ell+1)}{2}\,\mathsf{t}^\ell_{mn}.
\end{aligned}$$

2.4 Fourier Analysis on $SU(2)$

This section is a summary of the main results about Fourier coefficients and Representation Theory of $SU(2)=\mathbb{S}^3$. Recall that we have an explicit basis of $L^2(\mathbb{S}^3)$ consisting of functions $\{\mathsf{t}^\ell\}_{mn}$, $\ell\in\frac{1}{2}\mathbb{N}_0$, $-\ell\le m,n\le\ell$ with step one.

The Fourier coefficient of a function $f\in C^\infty(\mathbb{S}^3)$, at $\ell\in\frac{1}{2}\mathbb{N}_0$, is given by

$$\widehat{f}(\ell) \doteq \int_{\mathbb{S}^3} f(x)\mathsf{t}^\ell(x)^* dx \in \mathbb{C}^{(2\ell+1)\times(2\ell+1)}.$$

The Fourier series becomes

$$f(x) = \sum_{\ell\in\frac{1}{2}\mathbb{N}_0}(2\ell+1)\mathrm{Tr}\left(\mathsf{t}^\ell(x)\widehat{f}(\ell)\right),$$

with the Plancherel's identity assuming the form

$$\|f\|_{L^2(\mathbb{S}^3)} = \left(\sum_{\ell\in\frac{1}{2}\mathbb{N}_0}(2\ell+1)\|\widehat{f}(\ell)\|^2_{\mathtt{HS}}\right)^{1/2}, \tag{2.1}$$

where $\|\widehat{f}(\ell)\|^2_{\mathtt{HS}} = \mathrm{Tr}(\widehat{f}(\ell)\widehat{f}(\ell)^*)$ is the Hilbert–Schmidt norm of the matrix $\widehat{f}(\ell)$.

Smooth functions and distributions on $\mathbb{S}^3$ can be characterized in terms of their Fourier coefficients in the following way:

$$f\in C^\infty(\mathbb{S}^3) \iff \forall N\ \exists C_N>0 \text{ such that } \|\widehat{f}(\ell)\|_{\mathtt{HS}}\le C_N(1+\ell)^{-N}, \forall\ell\in\frac{1}{2}\mathbb{N}_0;$$

and,

$$u\in\mathcal{D}'(\mathbb{S}^3) \iff \exists M\in\mathbb{N}\ \exists C>0 \text{ such that } \|\widehat{u}(\ell)\|_{\mathtt{HS}}\le C(1+\ell)^M, \forall\ell\in\frac{1}{2}\mathbb{N}_0.$$

In the next chapter we are going to deal with operators on the product $\mathbb{T}^1 \times \mathbb{S}^3$ or more generally on $\mathbb{T}^1 \times (\mathbb{S}^3)^N$, so it will be important to state the main results about partial Fourier coefficients in these cases.

For each $f \in L^1(\mathbb{T}^1 \times \mathbb{S}^3)$ and $\ell \in \frac{1}{2}\mathbb{N}_0$, we define the mn–component of the partial Fourier coefficient of f with respect to the x variable as

$$\widehat{f}(t,\ell)_{mn} = \int_{\mathbb{S}^3} f(t,x)\overline{\mathrm{t}^\ell(x)_{nm}}\, dx \in L^1(\mathbb{T}^1),$$

and, for each $k \in \mathbb{Z}$, we denote by $\widehat{\widehat{f}}(k,\ell)_{mn}$ the k–th Fourier coefficient of the function $\widehat{f}(\cdot,\ell)_{mn}$.

We have the following characterizations of the spaces $C^\infty(\mathbb{T}^1 \times \mathbb{S}^3)$, $\mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$, and $C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$ (see [22] and [23]).

Proposition 2.8. *Let $\{\widehat{f}(\,\cdot\,,\ell)_{mn}\}$ be a sequence of functions on $\mathbb{T}^1$ and define*

$$f(t,x) \doteq \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) \sum_{m,n} \widehat{f}(t,\ell)_{mn} \mathrm{t}^\ell(x)_{nm}, \; (t,x) \in \mathbb{T}^1 \times \mathbb{S}^3.$$

Then $f \in C^\infty(\mathbb{T}^1 \times \mathbb{S}^3)$ if and only if $\widehat{f}(\,\cdot\,,\ell)_{mn} \in C^\infty(\mathbb{T}^1)$, for all $\ell \in \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m,n \leq \ell$ and for every multi-index β and $N > 0$ there exists $C_{\beta N} > 0$ such that

$$\left|\partial^\beta \widehat{f}(t,\ell)_{mn}\right| \leq C_{\beta N}(1+\ell)^{-N}, \quad \forall t \in \mathbb{T}^1, \; \ell \in \tfrac{1}{2}\mathbb{N}_0, \; -\ell \leq m,n \leq \ell.$$

Proposition 2.9. *Let $\left\{\widehat{u}(\,\cdot\,,\eta)_{rs}\right\}$ be a sequence of distributions on $\mathbb{T}^1$ and define*

$$u = \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) \sum_{m,n} \widehat{u}(\,\cdot\,,\eta)_{mn} \mathrm{t}^\ell_{nm}.$$

Then $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$ if and only if there are $K \in \mathbb{N}$ and $C > 0$ such that

$$\left|\langle \widehat{u}(\cdot,\ell)_{mn}, \varphi\rangle\right| \leq C\, p_K(\varphi)(\ell+1)^K, \tag{2.2}$$

for all $\varphi \in C^\infty(\mathbb{T}^1)$ and $\ell \in \frac{1}{2}\mathbb{N}_0$, where $p_K(\varphi) \doteq \sum_{\beta \leq K} \left\|\partial^\beta \varphi\right\|_{L^\infty(\mathbb{T}^1)}$.

Proposition 2.10. *Let $f \in C^\infty(\mathbb{T}^1 \times \mathbb{S}^3)$. We have that $f \in C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$ if and only if $\widehat{f}(\cdot,\ell)_{mn} \in C^\omega(\mathbb{T}^1)$ for every $\ell \in \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m,n \leq \ell$, and there are $C, B > 0$ such that*

$$|\widehat{f}(t,\ell)_{mn}| \leq Ce^{-B\ell},$$

for all $t \in \mathbb{T}^1$, $\ell \in \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m,n \leq \ell$.

Chapter 3

Invariant constant-coefficient operators

In this chapter, we present our contribution to the study of the global analytic hypoellipticity of a class of constant-coefficient operators on a general product of compact Lie groups $G_1 \times G_2$.

In order to characterize the Global Analytic Hypoellipticity of the class we just mentioned, in Section 3.1 we define the set of "null frequencies" and an analytic Diophantine condition. In Section 3.2 we introduce equivalent analytic Diophantine conditions for the special cases $\mathbb{T}^1 \times G_2$ and $\mathbb{T}^1 \times \mathbb{S}^3$.

3.1 Global analytic hypoellipticity for constant-coefficient operators

Let G_1 and G_2 be compact Lie groups, $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{g}_2$ and $\alpha, q \in \mathbb{C}$. Define

$$L = X_1 + \alpha X_2 + q.$$

Up to a product by a constant, this is the most general constant-coefficient left-invariant operator with a perturbation by a constant on the product $G_1 \times G_2$. Remarks about perturbations of this operator by functions will be done later. We aim to characterize completely the global analytic hypoellipticity of this class of operators.

Recall that for each $([\xi], [\eta]) \in \widehat{G} = \widehat{G}_1 \times \widehat{G}_2$, we can choose a matrix representative of $[\xi]$ and a matrix representative of $[\eta]$ such that

$$\sigma_{X_1}(\xi)_{mn} = i\lambda_m(\xi) \cdot \delta_{mn}, \text{ for } 1 \leq m, n \leq d_\xi$$

and

$$\sigma_{X_2}(\eta)_{rs} = i\mu_r(\eta) \cdot \delta_{rs}, \text{ for } 1 \leq r, s \leq d_\eta.$$

In this way, if $u \in \mathcal{D}'(G)$ satisfies $Lu = f \in \mathcal{D}'(G)$, then after taking two consecutive partial Fourier coefficients in this equation, once in each variable, we obtain the equations

$$\widehat{\widehat{Lu}}(\xi, \eta)_{mn_{rs}} = (i\lambda_m(\xi) + i\alpha\mu_r(\eta) + q)\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}} = \widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}},$$

for each $1 \leq m, n \leq d_\xi$ and $1 \leq r, s \leq d_\eta$. Note that if $(i\lambda_m(\xi) + i\alpha\mu_r(\eta) + q) = 0$, then $\widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}} = 0$ and if $(i\lambda_m(\xi) + i\alpha\mu_r(\eta) + q) \neq 0$, then

$$\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}} = (i\lambda_m(\xi) + i\alpha\mu_r(\eta) + q)^{-1}\widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}}.$$

So we are lead to consider the following set:

$$\mathcal{N} = \left\{([\xi], [\eta]) \in \widehat{G}; (\lambda_m(\xi) + \alpha\mu_r(\eta) - iq) = 0 \text{ for some } 1 \leq m \leq d_\xi, 1 \leq r \leq d_\eta\right\}.$$

Proposition 3.1. *If the set $\mathcal{N}$ is infinite, then there is a distribution $u \in \mathcal{D}'(G) \setminus C^\infty(G)$ such that $Lu = 0$. In particular, L is not (GH) (neither (GAH)).*

Proof. We define the sequence

$$\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}} = \begin{cases} 1 & \text{if } \lambda_m(\xi) + \alpha\mu_r(\eta) - iq = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Since $|\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}}| \leq 1$, it is clear that this sequence defines a distribution $u \in \mathcal{D}'(G)$, which by construction satisfies $Lu = 0$. Now, since $\mathcal{N}$ is not finite, $u \notin C^\infty(G)$ because its Fourier coefficients do not decay. □

Definition 3.2. *We say that L satisfies the (ADC) condition if for all $B > 0$ there is a constant $K > 0$ such that*

$$|\lambda_m(\xi) + \alpha\mu_r(\eta) - iq| \geq Ke^{-B(\langle\xi\rangle + \langle\eta\rangle)}$$

for all $([\xi], [\eta]) \in \widehat{G}$, $1 \leq m \leq d_\xi$, $1 \leq r \leq d_\eta$, such that $(i\lambda_m(\xi) + i\alpha\mu_r(\eta) + q) \neq 0$.

The above definition is a Diophantine condition-like for the operator L.

Proposition 3.3. *If L satisfies the (ADC) condition and $\mathcal{N}$ is finite, then L is (GAH).*

Proof. Let $u \in \mathcal{D}'(G)$ such that $Lu = f \in C^\omega(G)$. Since $f \in C^\omega(G)$ there are constants $K_1 > 0$ and $B_1 > 0$ such that

$$|\widehat{\widehat{f}}(\xi,\eta)_{mn_{rs}}| \leq K_1 e^{-B_1(\langle\xi\rangle+\langle\eta\rangle)}$$

for all $([\xi],[\eta]) \in \widehat{G}$ and indexes $1 \leq m,n \leq d_\xi$ and $1 \leq r,s \leq d_\eta$.
From the definition of (ADC) condition, if we choose $B > 0$ with $\widetilde{B} := B_1 - B > 0$, then there is a positive constant $K > 0$ such that for the indexes m,r with $i\lambda_m(\xi) + i\alpha\mu_r(\eta) + q \neq 0$ we have

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq K e^{B(\langle\xi\rangle+\langle\eta\rangle)} \cdot K_1 e^{-B_1(\langle\xi\rangle+\langle\eta\rangle)} = K \cdot K_1 e^{-(B_1-B)(\langle\xi\rangle+\langle\eta\rangle)} = \widetilde{K} e^{-\widetilde{B}(\langle\xi\rangle+\langle\eta\rangle)}.$$

By hypothesis, the set of the remaining indexes associated to representations $([\xi],[\eta]) \in \mathcal{N}$ is finite, so there is a constant $C > 0$ such that

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq C = C \cdot e^{\widetilde{B}(\langle\xi\rangle+\langle\eta\rangle)} \cdot e^{-\widetilde{B}(\langle\xi\rangle+\langle\eta\rangle)} \leq \widetilde{C} e^{-\widetilde{B}(\langle\xi\rangle+\langle\eta\rangle)}$$

for all $([\xi],[\eta]) \in \mathcal{N}$ and corresponding indexes m,n,r,s, where

$$\widetilde{C} = \max_{([\xi],[\eta])\in\mathcal{N}} \{C \cdot e^{\widetilde{B}(\langle\xi\rangle+\langle\eta\rangle)}\}.$$

So, if $K' = \max\{\widetilde{C}, \widetilde{K}\}$, then

$$|\widehat{\widehat{u}}(\xi,\eta)_{mn_{rs}}| \leq K' e^{-\widetilde{B}(\langle\xi\rangle+\langle\eta\rangle)}$$

for all $([\xi],[\eta]) \in \widehat{G}$ and corresponding indexes m,n,r,s. Then $u \in C^\omega(G)$ and L is (GAH). $\square$

Now we are going to prove a converse of the above proposition. We already know that if L is (GAH), then $\mathcal{N}$ must be finite. So we only need to prove that if L is (GAH), then L satisfies the (ADC) condition, which will be done in the below proposition. Proposition 3.3 together with Proposition 3.4 complete the characterization of when L is (GAH).

Proposition 3.4. *If L is (GAH), then it satisfies the (ADC) condition.*

Proof. If L does not satisfy the (ADC) condition, then there are a constant $\widetilde{B} > 0$ and a sequence of representations $([\xi_j],[\eta_j]) \in \widehat{G}$ such that

$$0 < |\lambda_{m_j}(\xi_j) + \alpha\mu_{r_j}(\eta_j) - iq| < \frac{1}{j} e^{-\widetilde{B}(\langle\xi_j\rangle+\langle\eta_j\rangle)},$$

for some $0 \leq m_j \leq d_{\xi_j}$ and $0 \leq r_j \leq d_{\eta_j}$ and for all $j \in \mathbb{N}$.

Observe that the set $\mathcal{A} = \{([\xi_j], [\eta_j]); j \in \mathbb{N}\}$ is infinite. Indeed, if it was not the case, since the right hand side of the above inequality goes to 0 when $j \to \infty$, the finite set of numbers in the middle term of the inequality would not be non zero for all big enough j.

Now, if we define

$$\widehat{\widehat{u}}(\xi, \eta)_{mn_{rs}} = \begin{cases} 1 & \text{for } m = m_j, r = r_j \text{ if } ([\xi, \eta]) = ([\xi_j, \eta_j]) \text{ for some } j \\ 0 & \text{otherwise} \end{cases},$$

then this sequence of coefficients clearly defines a distribution $u \in \mathcal{D}'(G) \setminus C^{\infty}(G)$.

We claim that $f := Lu \in C^{\omega}(G)$. If $([\xi, \eta]) \notin \mathcal{A}$, then $|\widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}}| = 0$ for all m, n, r, s. On the other hand, if $([\xi, \eta]) = ([\xi_j, \eta_j]) \in \mathcal{A}$, then

$$\begin{aligned} |\widehat{\widehat{f}}(\xi, \eta)_{m_j n_{r_j s}}| &= |\lambda_{m_j}(\xi_j) + \alpha \mu_{r_j}(\eta_j) - iq| \cdot |\widehat{\widehat{u}}(\xi, \eta)_{m_j n_{r_j s}}| \\ &\leq \frac{1}{j} e^{-\tilde{B}(\langle \xi_j \rangle + \langle \eta_j \rangle)} \\ &\leq e^{-\tilde{B}(\langle \xi_j \rangle + \langle \eta_j \rangle)}. \end{aligned}$$

for all $j \in \mathbb{N}$. In any case, we have

$$|\widehat{\widehat{f}}(\xi, \eta)_{mn_{rs}}| \leq e^{-\tilde{B}(\langle \xi \rangle + \langle \eta \rangle)}$$

for all $([\xi], [\eta]) \in \widehat{G}$ and corresponding indexes m, n, r, s, therefore $f \in C^{\omega}(G)$ and L is not (GAH). $\square$

We can summarize the above results in the following.

Theorem 3.5. *Let $\alpha, q \in \mathbb{C}$. The operator $L = X_1 + \alpha X_2 + q$ is (GAH) if and only if:*

1. *It satisfies the (ADC) condition;*
2. *The set $\mathcal{N}$ is finite.*

3.2 Special classes of operators and equivalent analytic Diophantine conditions

In the special case where $G_1 = \mathbb{T}^1$ and the operator L has the form

$$L = \partial_t + \alpha X + q,$$

with $\alpha, q \in \mathbb{C}$ and $X \in \mathfrak{g}_2$, then the (ADC) condition has an equivalent formulation given by the following lemma.

Lemma 3.6. *The operator $L = \partial_t + \alpha X + q$ on $G = \mathbb{T}^1 \times G_2$ satisfies (ADC) if and only if the following condition holds*
(ADC'): for all $B > 0$ there is a constant $C > 0$ such that

$$|1 - e^{\pm 2\pi(i\mu_r(\eta)\alpha + q)}| \geq Ce^{-B\langle \eta \rangle},$$

for all $[\eta] \in \widehat{G}_2$ and $1 \leq r \leq d_\eta$ such that $i\mu_r(\eta)\alpha + q \notin i\mathbb{Z}$.

Proof. If L does not satisfy the (ADC') condition, then there are a constant $B > 0$ and a sequence of representations $[\eta_j] \in \widehat{G}_2$ and indexes $1 \leq r_j \leq d_{\eta_j}$ such that

$$0 < |1 - e^{\pm 2\pi(i\mu_{r_j}(\eta_j)\alpha + q)}| < \frac{1}{j} e^{-B\langle \eta_j \rangle}$$

for all $j \in \mathbb{N}$. In particular,

$$e^{\pm 2\pi(i\mu_{\eta_j}\alpha + q)} = e^{2\pi(\mathrm{Re}(q) - \mu_{r_j}(\eta_j)\mathrm{Im}(\alpha))} \cdot e^{2\pi i(\mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q))} \to 1 \text{ when } j \to \infty,$$

so $|\mathrm{Re}(q) - \mu_{r_j}(\eta_j)\mathrm{Im}(\alpha)| \to 0$ and there is a sequence of integers (τ_j) such that

$$|\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)| \to 0,$$

when $j \to \infty$.
By the Mean Value Theorem (MVT) we can choose j large enough such that

$$\begin{aligned} |1 - e^{\pm 2\pi(i\mu_{r_j}(\eta_j)\alpha + q)}| \geq & |1 - e^{2\pi(\mathrm{Re}(q) - \mu_{r_j}(\eta_j)\mathrm{Im}(\alpha))}| \\ \geq & e^{-1} 2\pi |\mathrm{Re}(q) - \mu_{r_j}(\eta_j)\mathrm{Im}(\alpha)| \end{aligned}$$

and

$$|\sin\left(2\pi(\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q))\right)| \geq \pi|\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)|,$$

which implies that

$$\begin{aligned} \pi|\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)| &\leq |\sin\left(2\pi(\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q))\right)| \\ &\leq 2e^{2\pi(\mathrm{Re}(q) - \mu_{r_j}(\eta_j)\mathrm{Im}(\alpha))} \\ &\quad \times |\sin\left(2\pi(\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q))\right)| \\ &= 2|\mathrm{Im}(1 - e^{\pm 2\pi(i\mu_{r_j}(\eta_j)\alpha + q)})| \\ &\leq 2|1 - e^{\pm 2\pi(i\mu_{r_j}(\eta_j)\alpha + q)}|. \end{aligned}$$

So there is a positive constant C such that for j large enough we have

$$\begin{aligned}0 <&|\tau_j + \mu_{r_j}(\eta_j)\alpha - iq|\\ &\leq |\mathrm{Re}(q) - \mu_{r_j}(\eta_j)\mathrm{Im}(\alpha)| + |\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)|\\ &\leq \frac{C}{j}e^{-B\langle \eta_j\rangle}.\end{aligned}$$

This concludes that L does not satisfy the (ADC) condition.

Now suppose that L does not satisfy the (ADC) condition, so there is a positive constant $B > 0$, a sequence of representations $(\tau_j, [\eta_j])$ in $\mathbb{Z} \times \widehat{G}_2$ and indexes $1 \leq r_j \leq d_{\eta_j}$ such that

$$0 < |\tau_j + \alpha\mu_{r_j}(\eta_j) - iq| \leq \frac{1}{j}e^{-B(|\tau_j|+\langle \eta_j\rangle)}$$

for all $j \in \mathbb{N}$. In particular,

$$|\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)| \to 0 \text{ and } |\mu_{r_j}(\eta_j)\mathrm{Im}(\alpha) - \mathrm{Re}(q)| \to 0$$

when $j \to \infty$.

Therefore, taking j big enough we can apply MVT again and obtain a constant $C > 0$ such that

$$\begin{aligned}|1 - e^{\pm 2\pi(i\mu_{r_j}(\eta_j)\alpha+q)}| \;\leq\;& |1 - e^{\pm 2\pi(\mathrm{Re}(q)-\mu_{r_j}(\eta_j)\mathrm{Im}(\alpha))}\cos(2\pi(\mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)))|\\ &+|e^{\pm 2\pi(\mathrm{Re}(q)-\mu_{r_j}(\eta_j)\mathrm{Im}(\alpha))}|\cdot|\sin(2\pi(\mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)))|\\ \leq\;& |1 - \cos(2\pi(\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)))| +\\ &+|1 - e^{\pm 2\pi(\mathrm{Re}(q)-\mu_{r_j}(\eta_j)\mathrm{Im}(\alpha))}| +\\ &+e^{\pm 2\pi(\mathrm{Re}(q)-\mu_{r_j}(\eta_j)\mathrm{Im}(\alpha))}|\sin(2\pi(\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)))|\\ \leq\;& C(|\tau_j + \mu_{r_j}(\eta_j)\mathrm{Re}(\alpha) + \mathrm{Im}(q)| + |\mathrm{Re}(q) - \mu_{r_j}(\eta_j)\mathrm{Im}(\alpha)|)\\ \leq\;& Ce^{-B(|\tau_j|+\langle \eta_j\rangle)},\end{aligned}$$

and then L does not satisfy the (ADC') condition.

□

This formulation of the (ADC) condition will be used mainly for theoretical purposes, but let us give at least a first example using this equivalence. Although simple, the next example shows how the global analytic hypoellipticity behaves with respect to zero order perturbations. We will present other classes of operators later on.

Example 3.7. *Consider the case of the operator $L = \partial_t + q$ defined on $\mathbb{T}^1 \times G_2$, that is, $X = 0 \in \mathfrak{g}_2$. In this case, we have*

$$\mathcal{N} = \{(\tau, [\eta]) \in \mathbb{Z} \times \widehat{G}_2; \tau - iq = 0\}$$

so it is clear that if $iq \in \mathbb{Z}$, then $\mathcal{N}$ is infinite and it follows from Proposition 3.1 that L is not (GAH). On the other hand, if $iq \notin \mathbb{Z}$, then

$$0 < C = |1 - e^{2\pi q}| \geq Ce^{-B\langle\eta\rangle}$$

for all $[\eta] \in \widehat{G}_2$, and so L is (GAH).

3.2.1 Constant coefficient operators on $\mathbb{T}^1 \times \mathbb{S}^3$

Now, with respect to the constant-coefficient operator

$$L = \partial_t + c\partial_0 + q, \tag{3.1}$$

defined on $\mathbb{T}^1 \times \mathbb{S}^3$, we have the associated set of null frequencies

$$\mathcal{N} = \left\{(k, \ell) \in \mathbb{Z} \times \tfrac{1}{2}\mathbb{N}_0; k + cm - iq = 0, \text{ for some } -\ell \leq m \leq \ell, \ \ell - m \in \mathbb{N}_0\right\}.$$

Remark 3.8. *The set of null frequencies $\mathcal{N}$ associated with the operator (3.1) is finite if and only if it is empty.*

Notice that if the set $\mathcal{N}$ is not empty, then there exist $(k, \ell_0) \in \mathbb{Z} \times \frac{1}{2}\mathbb{N}_0$ and an index $-\ell_0 \leq m \leq \ell_0$ such that $k + cm - iq = 0$. But by the disposition of the indexes on the matrix representations t^ℓ, the same index m can be used with any other $\ell \geq \ell_0$. In other words, $(k, \ell) \in \mathcal{N}$ for all $\ell \geq \ell_0$, and therefore $\mathcal{N}$ is infinite. So when we suppose that $\mathcal{N}$ is finite, we are actually supposing that $\mathcal{N}$ is empty.

In the case of operators defined on $\mathbb{T}^1 \times \mathbb{S}^3$, there are more ways to formulate the (ADC) condition, which we are going to state in the next lemma. These are useful when we are dealing with examples.

Lemma 3.9. *Let $c, q \in \mathbb{C}$. The following Diophantine conditions are equivalent:*

(ADC) $\forall B > 0$, $\exists K_B > 0$ *such that*

$$|k + cm - iq| \geq K_B e^{-B(|k|+\ell)},$$

for all $k \in \mathbb{Z}$, $\ell \in \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m \leq \ell$, $\ell - m \in \mathbb{N}_0$, whenever $k + cm - iq \neq 0$.

(ADC2) $\forall B > 0$, $\exists K_B > 0$ *such that*

$$|k + c\ell - iq| \geq K_B e^{-B(|k|+|\ell|)},$$

for all $k \in \mathbb{Z}$, $\ell \in \frac{1}{2}\mathbb{Z}$, *whenever* $k + c\ell - iq \neq 0$.

(ADC3) $\forall B > 0$, $\exists K_B > 0$ *such that*

$$\left|k + \tfrac{c}{2}\ell - iq\right| \geq K_B e^{-B(|k|+|\ell|)},$$

for all $k, \ell \in \mathbb{Z}$, *whenever* $k + \frac{c}{2}\ell - iq \neq 0$.

Proof. First, let us prove the equivalence between (ADC) and (ADC2). Assume that (ADC) holds and let $\tau \in \mathbb{Z}$, $\rho \in \frac{1}{2}\mathbb{Z}$ such that $\tau + c\rho - iq \neq 0$. Take $k = \tau$, $\ell = |\rho|$, and $m = \rho$ in (ADC). Hence,

$$|\tau + c\rho - iq| \geq K_B e^{-B(|\tau|+|\rho|)}.$$

Assume now that (ADC2) holds and let $\tau \in \mathbb{Z}$, $\rho \in \frac{1}{2}\mathbb{N}_0$, and $-\rho \leq m \leq \rho$ such that $\rho - m \in \mathbb{N}_0$ and $\tau + cm - iq \neq 0$. Take $k = \tau$ and $\ell = m$ in (ADC2). So, since $|m| \leq \rho$, we have

$$|\tau + cm - iq| \geq K_B e^{-B(|\tau|+|m|)} \geq K_B e^{-B(|\tau|+\rho)},$$

which concludes the proof of the equivalence (ADC) $\iff$ (ADC2). Now, assume the validity of (ADC2) and let $\tau, \rho \in \mathbb{Z}$ such that $k + \frac{c}{2}\rho - iq \neq 0$. Take $k = \tau$ and $\ell = \frac{\rho}{2}$ in (ADC2) . So,

$$|\tau + \tfrac{c}{2}\rho - iq| \geq K_B e^{-B(|\tau|+|\rho|/2)} \geq K_B e^{-B(|\tau|+|\rho|)},$$

which implies that condition (ADC3) holds. Finally, assume that (ADC3) holds and let $\tau \in \mathbb{Z}$, $\rho \in \frac{1}{2}\mathbb{Z}$ such that $\tau + c\ell - iq \neq 0$. We can write $\rho = \frac{r}{2}$, for some $r \in \mathbb{Z}$. Take $k = \tau$ and $\ell = r$ in (ADC3) . Thus,

$$|\tau + c\rho - iq| = |\tau + \tfrac{c}{2}r - iq| \geq K_B e^{-B(|k|+|r|)} \geq K_B e^{-2B(|k|+|\rho|)},$$

and, by adjusting B, we have the validity of (ADC2) . $\square$

Example 3.10. *Let* $c, q \in \mathbb{C}$ *and consider the operator* $L = \partial_t + c\partial_0 + q$ *defined on* $\mathbb{T}^1 \times \mathbb{S}^3$. *Writing* $c = a + ib$, *with* $a, b \in \mathbb{R}$, *we have*

$$k + cm - iq = (k + am + \mathrm{Im}(q)) + i(mb - \mathrm{Re}(q)).$$

Thus, if $b \neq 0$ *and* $\mathrm{Re}(q)/b \notin \frac{1}{2}\mathbb{Z}$, *then*

$$|k + cm - iq| \geq |b(m - \mathrm{Re}(q)/b)| \geq K = \textit{constant}.$$

Hence, the set $\mathcal{N}$ is empty and the condition (ADC) is satisfied, then by Proposition 3.5 we have that L is (GAH).
If $b \neq 0$ and $\mathrm{Re}(q)/b \in \frac{1}{2}\mathbb{Z}$ we have two cases to consider.

When $\mathrm{Im}(q) + \mathrm{Re}(q)a/b \in \mathbb{Z}$ the set $\mathcal{N}$ has infinitely many elements, hence L is not (GAH) by Proposition 3.1.
When $\mathrm{Im}(q) + \mathrm{Re}(q)a/b \notin \mathbb{Z}$, the set $\mathcal{N}$ is empty and $\delta = \inf\limits_{k\in\mathbb{Z}}\{k + \mathrm{Re}(q)a/b + \mathrm{Im}(q)\} > 0$. Hence,

$$|k + cm - iq| \geq \max\{|b|, \delta\} = \textit{constant} > 0,$$

which implies that L is (GAH).
Similarly, if $b = 0$ and $\mathrm{Re}(q) \neq 0$ we obtain

$$|k + cm - iq| \geq |\mathrm{Re}(q)| \geq K = \textit{constant},$$

and, consequently L is (GAH).

Now, let us deal with the case where $\mathrm{Re}(q) = b = 0$. By the condition (ADC3) the global analytic hypoellipticity of L depends on the approximations $\left|k + \frac{a}{2}\ell + \mathrm{Im}(q)\right|$, with $k, \ell \in \mathbb{Z}$. In the special case where $b = 0$ and $iq \in \mathbb{Z} + \frac{a}{2}\mathbb{Z}$, then $\left|k + \frac{c}{2}\ell - iq\right| = \left|\widetilde{k} + \frac{a}{2}\widetilde{\ell}\right|$, with $\widetilde{k}, \widetilde{\ell} \in \mathbb{Z}$. Therefore, the operator

$$L = \partial_t + a\partial_0 + q \textit{ is } (GAH) \Leftrightarrow a \textit{ is not exponential Liouville}.$$

Finally, when $\mathrm{Re}(q) = b = 0$ and $iq \notin \mathbb{Z} + \frac{a}{2}\mathbb{Z}$, then the set $\mathcal{N}$ is empty. Moreover, if $a \in \mathbb{Q}$, then we can also guarantee the existence of a constant $K > 0$ such that

$$\left|k + \tfrac{a}{2}\ell + \mathrm{Im}(q)\right| \geq K = \textit{constant},$$

since in this case the set $\mathbb{Z} + \frac{a}{2}\mathbb{Z}$ is discrete. Then the condition (ADC3) is satisfied, which implies that L is (GAH). On the other hand, for $a \in \mathbb{R}\setminus\mathbb{Q}$ the set $\mathbb{Z} + \frac{a}{2}\mathbb{Z}$ is dense in $\mathbb{R}$, then L is (GAH) if and only if the condition (ADC3) is satisfied.

Chapter 4

A class of invariant evolution operators

Inspired by the works of Hounie [20, 21], Bergamasco [2, 3], Petronilho [24] and so many other researchers that have studied global properties of vector fields on many classes of manifolds, in particular the works on tori, it is natural to inquire if the operator

$$\partial_t + c(t)\partial_0,$$

is globally analytic hypoelliptic on $\mathbb{T}^1 \times \mathbb{S}^3$, when $c \in C^\omega(\mathbb{T}^1)$.

Let us prove that the answer to this question is negative. For $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$ and $\ell \in \frac{1}{2}\mathbb{N}_0$, the matrix entries of the Fourier partial coefficient of $[\partial_t + c(t)\partial_0](u)$ are given by

$$[\partial_t + imc(t)]\, \widehat{u}(t,\ell)_{mn}, \text{ for } -\ell \leq m, n \leq \ell.$$

Now consider the sequence of real analytic functions $\{\widehat{u}(t,\ell); \ell \in \frac{1}{2}\mathbb{N}_0\}$ defined by

$$\widehat{u}(t,\ell)_{mn} = \begin{cases} 1, & \text{whenever } m = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly this sequence defines $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$ such that $(\partial_t + c(t)\partial_0)u = 0$. Moreover, since the set $\{\ell \in \frac{1}{2}\mathbb{N}_0; \widehat{u}(t,\ell)_{mn} = 1\}$ has infinitely many elements, this sequence does not correspond to any real analytic function, finishing the proof.

However, recall that in the examples 3.7 and 3.10 we present some interesting cases of globally analytic hypoelliptic operators by adding a zero-order perturbation to a constant-coefficient vector field defined on $\mathbb{T}^1 \times \mathbb{S}^3$. Therefore it seems reasonable to consider a class of perturbed operators of the form $\partial_t + c(t)\partial_0 + q$, with $q \in \mathbb{C}$.

4.1 A necessary condition to the global analytic hypoellipticity

Consider the following operator on $\mathbb{T}^1 \times \mathbb{S}^3$,

$$P := \partial_t + c(t)\partial_0 + q, \tag{4.1}$$

where $q \in \mathbb{C}$ and $c(t) = a(t) + ib(t)$, with $a, b \in C^\omega(\mathbb{T}^1, \mathbb{R})$.

In the remainder of this work we will also adopt some notations. Define the function $C(t) = A(t) + iB(t)$, where

$$A(t) = \int_0^t a(s)ds \text{ and } B(t) = \int_0^t b(s)ds,$$

and $c_0 = a_0 + ib_0$, where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} a(s)ds \text{ and } b_0 = \frac{1}{2\pi}\int_0^{2\pi} b(s)ds. \tag{4.2}$$

As in the case of the torus, we will see that the answer of when P is (GAH) is related with the same one about a constant coefficient operator associated with P. Then, let us also define the operator

$$P_0 := \partial_t + c_0\partial_0 + q. \tag{4.3}$$

The next proposition establishes a necessary condition for the global analytic hypoellipticity of the operator P, similar to Proposition 3.1. For that, we are lead to consider the set

$$\mathcal{N}_0 = \left\{(k, \ell) \in \mathbb{Z} \times \tfrac{1}{2}\mathbb{N}_0; k + c_0 m - iq = 0, \text{ for some } -\ell \leq m \leq \ell,\ \ell - m \in \mathbb{N}_0\right\},$$

which is exactly the same set associated with the operator P_0. Note that this set has a clear correspondence with the set

$$\left\{\ell \in \frac{1}{2}\mathbb{N}_0; imc_0 + q \in i\mathbb{Z}, \text{ for some } -\ell \leq m \leq \ell,\ \ell - m \in \mathbb{N}_0\right\}.$$

Proposition 4.1. *If the operator P defined in (4.1) is (GAH), then $\mathcal{N}_0$ is finite.*

Proof. Suppose by contradiction that $\mathcal{N}_0$ has infinitely many elements. In this case, there are sequences $\{(k_j, \ell_j)\}_{j\in\mathbb{N}}$ in $\mathbb{Z} \times \frac{1}{2}\mathbb{N}$ and $\{m_j\}_{j\in\mathbb{N}}$ in $\frac{1}{2}\mathbb{Z}$, with $-\ell_j \leq m_j \leq \ell_j$, such that

$$k_j + c_0 m_j - iq = 0,\ j \in \mathbb{N}. \tag{4.4}$$

For each $j \in \mathbb{N}$, let $t_j \in [0, 2\pi]$ and $M_j \in \mathbb{R}$ such that

$$M_j = \int_0^{t_j} \big[\mathrm{Re}(q) - m_j b(\sigma)\big] d\sigma = \max_{0 \le t \le 2\pi} \int_0^{t} \big[\mathrm{Re}(q) - m_j b(\sigma)\big] d\sigma,$$

and consider the sequence

$$\widehat{u}(t,\ell)_{mn} = \begin{cases} \exp\left\{ \displaystyle\int_0^t \big[i m_j c(\sigma) + q\big] d\sigma - M_j \right\}, & \text{if } \ell = \ell_j \text{ and } m = n = m_j \\ 0 & \text{otherwise.} \end{cases}$$

Since $im_jc_0 + q \in i\mathbb{Z}$, for all $j \in \mathbb{N}$, all the partial Fourier coefficients $\widehat{u}(\cdot, \ell) \in C^\infty(\mathbb{T}^1)$ and

$$|\widehat{u}(t,\ell_j)_{m_j m_j}| = \exp\left\{ \int_0^t \big[\mathrm{Re}(q) - m_j b(\sigma)\big] - M_j \right\} \le 1, \text{ for all } t \in [0, 2\pi],$$

therefore the sequence $\{\widehat{u}(\cdot,\ell); \ell \in \frac{1}{2}\mathbb{N}\}$ correspond to a distribution $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$, moreover

$$\widehat{u}(t_j,\ell_j)_{m_j m_j} = 1, \ j \in \mathbb{N}.$$

Now note that set $\{\ell_j\}_{j\in\mathbb{N}}$ cannot be bounded. Indeed, if it were bounded, by (4.4), the set $\{k_j\}_{j\in\mathbb{N}}$ would also be bounded and, consequently, the set $\mathcal{N}_0$ would be finite (because $\mathcal{N}_0$ is discrete). Hence, $\{\ell_j\}_{j\in\mathbb{N}}$ has infinitely many elements and $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3) \setminus C^\infty(\mathbb{T}^1 \times \mathbb{S}^3)$. Since

$$[\partial_t + i m_j c(t) + q]\widehat{u}(t,\ell_j)_{m_j m_j} = 0, \ j \in \mathbb{N},$$

then $Pu = 0$ and P is not (GAH). □

Motivated by Proposition 4.1, our next step will be to find conditions on q and $c(t)$ such that $\mathcal{N}_0$ is finite.

Suppose that $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$ is a solution of $Pu = f \in C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$, then for each $\ell \in \frac{1}{2}\mathbb{N}$ and $-\ell \le m, n \le \ell$ we have

$$\widehat{Pu}(t,\ell)_{mn} = \big[\partial_t + imc(t) + q\big]\widehat{u}(t,\ell)_{mn} = \widehat{f}(t,\ell)_{mn},$$

that is equivalent to

$$\big[\partial_t + imc_0 + q\big](e^{im\mathscr{C}(t)}\widehat{u}(t,\ell)_{mn}) = e^{im\mathscr{C}(t)}\widehat{f}(t,\ell)_{mn}, \tag{4.5}$$

where

$$\mathscr{C}(t) \doteq C(t) - c_0 t.$$

All solutions of the ordinary differential equations (4.5) are analytical and their expressions depend on the value

$$\gamma_m = imc_0 + q = (\mathrm{Re}(q) - mb_0) + i(ma_0 + \mathrm{Im}(q)). \tag{4.6}$$

The next result gives us the form of a solution in terms of a fixed constant.

Lemma 4.2. *Let $\gamma \in \mathbb{C}$, $g \in C^\infty(\mathbb{T}^1)$, and consider the equation*

$$\frac{d}{dt}v(t) + \gamma v(t) = g(t). \tag{4.7}$$

If $\gamma \notin i\mathbb{Z}$ then the equation (4.7) has a unique solution that can be expressed by

$$v(t) = \frac{1}{1 - e^{-2\pi\gamma}} \int_0^{2\pi} e^{-\gamma s} g(t - s)\, ds, \tag{4.8}$$

or equivalently,

$$v(t) = \frac{1}{e^{2\pi\gamma} - 1} \int_0^{2\pi} e^{\gamma r} g(t + r)\, dr. \tag{4.9}$$

If $\gamma \in i\mathbb{Z}$ and $\int_0^{2\pi} e^{\gamma s} g(s)\, ds = 0$ then we have that

$$v(t) = e^{-\gamma t} \int_0^t e^{\gamma s} g(s)\, ds \tag{4.10}$$

is a solution of the equation (4.7).

In order to prove Lemma 4.2 is sufficient to note that $E = (1 - e^{-2\pi\gamma})^{-1} e^{\gamma t}$ is the fundamental solution of the operator $d/dt + \gamma$, when $\gamma \notin i\mathbb{Z}$. The equivalence between (4.8) and (4.9) follows from the change of variable $s \mapsto -r + 2\pi$. We point out that the solution (4.10) is not unique. As a final observation about this lemma, note that if $g \in C^\omega(\mathbb{T}^1)$, then the solutions of (4.7) are also real analytic.

Turning back to the problem of determining under what conditions the set $\mathcal{N}_0$ is finite, recall that, by Remark 3.8, if $\mathcal{N}_0$ is finite then $\mathcal{N}_0 = \varnothing$ and $\gamma_m \notin i\mathbb{Z}$ for all m.

We are going to explore what is the algebraic meaning of that. Notice that there exist $\ell \in \frac{1}{2}\mathbb{N}_0$ and $-\ell \leq m \leq \ell$ such that $\gamma_m \in i\mathbb{Z}$ if and only if there exist ℓ and m such that $\mathrm{Re}(q) - mb_0 = 0$ and $\mathrm{Im}(q) + ma_0 \in \mathbb{Z}$. If $b_0 \neq 0$, then this is equivalent to $m = \frac{\mathrm{Re}(q)}{b_0} \in \frac{1}{2}\mathbb{Z}$ and $\mathrm{Im}(q) + \mathrm{Re}(q)\frac{a_0}{b_0} \in \mathbb{Z}$. If $b_0 = 0$, then $\gamma_m \in i\mathbb{Z}$ for some ℓ and m is equivalent to $\mathrm{Re}(q) = 0$ and $\mathrm{Im}(q) \in \mathbb{Z} + \frac{a_0}{2}\mathbb{Z}$.

So let us define the following algebraic conditions:

(C1) $b_0 \neq 0$ and either $\frac{\mathrm{Re}(q)}{b_0} \notin \frac{1}{2}\mathbb{Z}$ or $\mathrm{Im}(q) + \mathrm{Re}(q)\frac{a_0}{b_0} \notin \mathbb{Z}$.

(C2) $b_0 = 0$ and either $\mathrm{Re}(q) \neq 0$ or $\mathrm{Im}(q) \notin \mathbb{Z} + \frac{a_0}{2}\mathbb{Z}$.

We conclude that $\gamma_m \notin i\mathbb{Z}$ for all m if and only if (C1) or (C2) holds.

4.2 The Nirenberg-Treves condition $(\mathcal{P})$

In this section we state our first result about global analytic hypoellipticity of operators with variable coefficients defined on $\mathbb{T}^1 \times \mathbb{S}^3$. We will begin by addressing the case where $b(t) = \mathrm{Im}\, c(t)$ does not change sign, that is, in the case where the Nirenberg-Treves condition $(\mathcal{P})$ holds.

Theorem 4.3. *Let $q \in \mathbb{C}$, $c(t) \in C^\omega(\mathbb{T}^1)$ and set $c_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} c(s)ds$. If*

1. $\mathrm{Im}\, c(t) \not\equiv 0$ *and does not change sign;*

2. $P_0 = \partial_t + c_0\partial_0 + q$ *satisfies the (ADC) condition;*

3. $q \in \mathbb{C}$ *satisfies the condition (C1).*

Then $P = \partial_t + c(t)\partial_0 + q$ is (GAH).

Proof. Writing $c(t) = a(t) + ib(t)$ and $c_0 = a_0 + ib_0$, since b does not change sign and is not identically zero, then $b_0 > 0$ or $b_0 < 0$. Up to a change of variable, we can assume without loss of generality that $b_0 < 0$.

By condition (C1), for each $\ell \in \frac{1}{2}\mathbb{N}$ and $-\ell \leq m, n \leq \ell$, the equation (4.5) has only one periodic real-analytic solution, which can be written in the two following equivalent ways:

$$\widehat{u}(t,\ell)_{mn} = (1 - e^{-2\pi\gamma_m})^{-1} \int_0^{2\pi} e^{-qs} e^{-im\int_{t-s}^{t} c(\sigma)d\sigma} \widehat{f}(t-s,\ell)_{mn} ds, \tag{4.11}$$

or

$$\widehat{u}(t,\ell)_{mn} = (e^{2\pi\gamma_m} - 1)^{-1} \int_0^{2\pi} e^{qs} e^{-im\int_{t}^{t+s} c(\sigma)d\sigma} \widehat{f}(t+s,\ell)_{mn} ds. \tag{4.12}$$

These expressions can be obtained from Lemma 4.2.

Since the two equivalent expressions above provide unique solutions to the ordinary differential equations (4.5), from this point the proof consists in to prove that these partial

Fourier coefficients satisfy the condition of decay at infinity given by Proposition 2.10 and Remark 1.43.

Let us make estimates for the case where $m \geq 0$, and for that we are going to use the expression (4.11). The other case is completely analogous and follows the same steps.

Since we are assuming $b_0 < 0$ and b does not change sign, then $b(\sigma) \leq 0$, for all $\sigma \in [0, 2\pi]$.

Moreover, from $m \geq 0$, we have

$$\left|\exp\left(-im\int_{t-s}^{t} c(\sigma)d\sigma\right)\right| = \exp\left(m\int_{t-s}^{t} b(\sigma)d\sigma\right) \leq 1, \quad \text{for all } t, s \in [0, 2\pi].$$

Recall that $f \in C^{\omega}(\mathbb{T}^1 \times \mathbb{S}^3)$, therefore by Proposition 2.10 there are constants $B > 0$ and $M > 0$ such that

$$|\widehat{f}(\cdot, \ell)_{mn}| \leq Me^{-B\ell},$$

for all $\ell \in \frac{1}{2}\mathbb{N}_0$ and $-\ell \leq m, n \leq \ell$.

Now, if $C > 0$ is such that $|e^{qs}| < C$, for all $s \in [0, 2\pi]$, then we have

$$\begin{aligned} |\widehat{u}(t, \ell)_{mn}| &\leq \left|e^{2\pi\gamma_m} - 1\right|^{-1} \int_0^{2\pi} |e^{qs}| \left|\exp\left(-im\int_t^{t+s} c(\sigma)d\sigma\right)\right| \left|\widehat{f}(t+s, \ell)_{mn}\right| ds \\ &\leq 2\pi MCe^{-B\ell}\left|e^{2\pi\gamma_m} - 1\right|^{-1}, \end{aligned}$$

for all $\ell \in \frac{1}{2}\mathbb{N}_0$ and $-\ell \leq m, n \leq \ell$.

Finally, since P_0 satisfies condition (ADC), choosing $\widetilde{B} > B$, by Lemma 3.6, there is a constant $\widetilde{M} > 0$ such that

$$|1 - e^{-2\pi\gamma_m}| \geq \widetilde{M}e^{-\widetilde{B}\ell},$$

for all $\ell \in \frac{1}{2}\mathbb{N}_0$, $-\ell \leq m \leq \ell$, $\ell - m \in \mathbb{N}_0$ such that $\gamma_m \notin i\mathbb{Z}$.

Thus

$$|\widehat{u}(t, \ell)_{mn}| \leq Ke^{-(\widetilde{B}-B)\ell},$$

for all $t \in [0, 2\pi]$, $\ell \in \frac{1}{2}\mathbb{N}_0$ and $-\ell \leq m, n \leq \ell$ with $m \geq 0$.

It follows from Proposition 2.10 and Remark 1.43 that $u \in C^{\omega}(\mathbb{T}^1 \times \mathbb{S}^3)$ and therefore P is (GAH). □

Example 4.4. *Let $n \neq 0$ be an integer number and set $b(t) = \sin(t) + n$. Then $b \in C^{\omega}(\mathbb{T}^1)$ does not change sign and $b_0 = n$. Taking $q \in \mathbb{C}$ with $\mathrm{Re}(q) \notin \mathbb{Q}$, there is $C > 0$ such that*

$|b_0 m - \text{Re}(q)| \geq C > 0$ *for all* $m \in \frac{1}{2}\mathbb{Z}$, *thus for any real valued analytic function* $a(t)$ *we have*

$$|(k + a_0 m + \text{Im}(q)) + i(b_0 m - \text{Re}(q))| \geq |b_0 m - \text{Re}(q)| \geq C.$$

Therefore P_0 *satisfies the condition (ADC) and* q *satisfies (C1). It follows from Theorem 4.3 that* $P = \partial_t + \big(a(t) + i(\sin(t) + n)\big)\partial_0 + q$ *is (GAH).*

What happens in the case where $b \equiv 0$? We will see in the next proposition that in this case P is analytically conjugated with P_0. This result is a particular case of a much more general proposition, whose reference the reader will find in the idea of the proof below. However, we are going to state just the case of our interest.

Proposition 4.5. *If* $b \equiv 0$, *then* P *is (GAH) if and only if* P_0 *is (GAH).*

Proof. If $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$ and $Pu = f$, then

$$[\partial_t + ima_0 + q](e^{im\mathscr{A}(t)}\widehat{u}(\,\cdot\,,\ell)_{mn}) = e^{im\mathscr{A}(t)}\widehat{f}(\,\cdot\,,\ell)_{mn} \tag{4.13}$$

where

$$\mathscr{A}(t) \doteq \int_0^t a(s)ds - a_0 t.$$

The correspondence between sequences of Fourier coefficients

$$\Psi_a\{\widehat{u}(\,\cdot\,,\ell)_{mn}\} = \{e^{im\mathscr{A}(t)}\widehat{u}(\,\cdot\,,\ell)_{mn}\} \tag{4.14}$$

defines an automorphism of $\mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$ (see Proposition 4.6 of [23]). Since $\mathscr{A}(t) \in \mathbb{R}$, we have $|e^{im\mathscr{A}(t)}| = 1$, which implies that Ψ_a is also an automorphism of $C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$. By (4.13) we have

$$P_0 \circ \Psi_a = \Psi_a \circ P,$$

so now it is clear that P is (GAH) if and only if P_0 is (GAH). □

4.3 Singular solutions

Let us consider the missing case: $b \not\equiv 0$ and b changes sign. We will prove that P is not (GAH) following the technique of building singular solutions introduced by A. Bergamasco and used by several authors, see for example [2, 4, 7, 11, 12, 13].

In our case, "to build a singular solution" means to present a suitable real analytic function f and a distributional solution u of $Pu = f$ that it is not a real analytic function. The main difference between our construction and that used in most of the references previously cited is that we cannot use cut-off functions, which makes the construction process more delicate.

To start building the singular solution recall that: the set of zeros of a not identically zero real analytic function are isolated and b is a 2π-periodic function that changes sign. Therefore we can assume, without loss of generality, that b changes sign from minus to plus at the point $t_0 = 0$ and that $b_0 \leq 0$.

Thus b is strictly positive on some open interval $]0, s[$ and there is $t^* \in \,]0, 2\pi[$ such that

$$M \doteq B(t^*) = \max_{t \in [0,2\pi]} B(t) > 0.$$

Define a 2π-periodic real analytic function by

$$\psi(t) = M + K(1 - \cos(t)) + i(a(0)\sin(t) - A(t^*)),$$

where $K > 0$ is a constant that we will chosen later on.

Proposition 4.6. *Let* $d_\ell = \left(1 - e^{-2\pi(i\ell c_0 + q)}\right)$, $\ell \in \frac{1}{2}\mathbb{N}$, *and consider the sequence of functions*

$$\widehat{f}(t,\ell)_{mn} = \begin{cases} d_\ell e^{-\ell\psi(t)}, & \textit{if } m = n = \ell; \\ 0 & \textit{otherwise.} \end{cases} \tag{4.15}$$

Then $\{\widehat{f}(\cdot,\ell)_{mn}\}$ *is the sequence of partial Fourier coefficients of a real analytic function on* $\mathbb{T}^1 \times \mathbb{S}^3$.

Proof. Notice that

$$|d_\ell| \leq 1 + e^{-2\pi(\mathrm{Re}(q) - \ell b_0)} = 1 + e^{-2\pi\mathrm{Re}(q)}e^{\ell 2\pi b_0} \leq 1 + e^{-2\pi\mathrm{Re}(q)} = C,$$

thus

$$|\widehat{f}(t,\ell)_{\ell\ell}| = |d_\ell| e^{-\ell\mathrm{Re}(\psi(t))} \leq Ce^{-\ell(M + K(1-\cos(t)))} \leq Ce^{-M\cdot\ell},$$

and all the functions $\widehat{f}(\cdot,\ell)_{mn} \in C^\omega(\mathbb{T}^1)$. It follows from Proposition 2.10 that the sequence $\{\widehat{f}(t,\ell)_{mn}\}$ defines $f \in C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$. □

The next step will be to construct a distribution $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3) \setminus C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$ satisfying $Pu = f$.

By Remark 3.8 and Proposition 3.1, we can assume that $\mathcal{N}_0 = \varnothing$. Thus, when $m = n = \ell$, the equation

$$[\partial_t + imc_0 + q](e^{im\mathscr{C}(t)}\widehat{u}(\,\cdot\,,\ell)_{mn}) = e^{im\mathscr{C}(t)}\widehat{f}(\,\cdot\,,\ell)_{mn}$$

has exactly only one real-analytic periodic solution, which is given by

$$\begin{aligned}\widehat{u}(t,\ell)_{\ell\ell} &= d_\ell^{-1}\int_0^{2\pi} e^{-qs}e^{-i\ell\int_{t-s}^{t}c(\sigma)d\sigma}\widehat{f}(t-s,\ell)_{\ell\ell}ds\\ &= \int_0^{2\pi} e^{-qs}e^{-\ell[\psi(t-s)+i(C(t)-C(t-s))]}ds.\end{aligned}$$

Let us denote $\Phi(t,s) \doteq \psi(t-s) + i(C(t) - C(t-s))$ and

$$\varphi(t,s) \doteq -\mathrm{Re}(\Phi(t,s)) = B(t) - B(t-s) - M - K(1-\cos(t-s)).$$

Lemma 4.7. *There exists a constant $K > 0$ such that $\varphi(t,s) \leq 0$, for all $t, s \in [0,2\pi]$.*

Proof. We will split this proof in three steps.

Step 1: There is $\delta_2 > 0$ such that $\varphi(t,s) \leq 0$ for all $t,s \in [0,2\pi]$ with $2\pi - \delta_2 < |t-s| \leq 2\pi$. We have $|t-s| = 2\pi$ if and only if $(t,s) = (2\pi,0)$ or $(t,s) = (0,2\pi)$. Since

$$\begin{aligned}&\varphi(2\pi,0) = B(2\pi) - B(2\pi) - M - K(1-\cos(2\pi)) = -M < 0, \text{ and}\\ &\varphi(0,2\pi) = B(0) - B(-2\pi) - M - K(1-\cos(-2\pi)) = 2\pi b_0 - M < 0,\end{aligned}$$

the desired result follows by continuity.

Step 2: There is $\delta_1 > 0$ such that $\varphi(t,s) \leq 0$ for all $t,s \in [0,2\pi]$ with $|t-s| < \delta_1$. For $t = s$ we have

$$\varphi(t,t) = B(t) - B(0) - M - K(1-\cos(0)) = B(t) - M \leq 0,$$

and for $t \neq s$ in $[0,2\pi]$ we have

$$\varphi(t,s) \leq 0 \iff \frac{B(t) - B(t-s) - M}{1-\cos(t-s)} \leq K.$$

Let us prove that for each fixed $t \in [0,2\pi]$ the function

$$g(u) \doteq \frac{B(t) - B(u) - M}{1-\cos(u)}$$

has an upper bound on some neighborhood of $u = 0$, which does not depend on t.

Since $B(t) - M \leq 0$, for all t, $1 - \cos(u) > 0$, for $u \neq 0$, $B(0) = 0$ and $B'(0) = b(0) = 0$ then by Taylor's formula we have

$$\begin{aligned} g(u) \leq & \frac{B(u)}{1-\cos(u)} \\ = & \frac{1}{1-\cos(u)}\left[B(0) + B'(0)u + \frac{B''(0)}{2}u^2 + R_2(u)\right] \\ = & \frac{B''(0)}{2}\frac{u^2}{1-\cos(u)} + \frac{R_2(u)}{1-\cos(u)} \end{aligned}$$

with $\lim_{u\to 0} R_2(u)/u^2 = 0$. Since

$$\lim_{u\to 0}\frac{u^2}{1-\cos(u)} = 2, \text{ and } \lim_{u\to 0}\frac{R_2(u)}{1-\cos(u)} = \lim_{u\to 0}\frac{R_2(u)}{u^2}\cdot\frac{u^2}{1-\cos(u)} = 0,$$

hence, given $K_1 > B''(0)$, there is $\delta_1 > 0$ such that $g(u) \leq K_1$ if $|u| < \delta_1$. Therefore

$$\varphi(t,s) = [B(t) - B(t-s) - M - K_1(1-\cos(t-s))] \leq 0, \text{ if } |t-s| < \delta_1.$$

Step 3: There is $K > K_1$ such that $\varphi(t,s) \leq 0$, for all $t, s \in [0, 2\pi]$ with $\delta_1 \leq |t-s| \leq 2\pi - \delta_2$. Notice that $1 - \cos(t-s) > 0$, for all $(t,s) \in R = \{(t,s) \in [0,2\pi]^2; \delta_1 \leq |t-s| \leq 2\pi - \delta_2\}$, and

$$\rho \doteq \min\{1 - \cos(u); \delta_1 \leq u \leq 2\pi - \delta_2\} > 0.$$

Given any $K \geq K_1$, we have that, for any $(t,s) \in R$,

$$\varphi(t,s) \leq B(t) - B(t-s) - \rho K \leq 0 \Leftrightarrow K \geq \frac{B(t) - B(t-s)}{\rho}.$$

Now, if necessary, we take a large K to obtain the last inequality.

□

Proposition 4.8. *The sequence*

$$\widehat{u}(t,\ell)_{mn} = \begin{cases} \displaystyle\int_0^{2\pi} e^{-qs}e^{-\ell\Phi(t,s)}ds, & \text{if } m = n = \ell; \\ 0 & \text{otherwise}, \end{cases} \tag{4.16}$$

with

$$\Phi(t,s) \doteq \psi(t-s) + i(C(t) - C(t-s)), \text{ for } t, s \in [0, 2\pi],$$

corresponds to a distribution $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$ that solves the equation $Pu = f$, where $f \in C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$ is defined in (4.15).

Proof. By Lemma 4.7, $\varphi(t,s) = -\mathrm{Re}(\Phi(t,s)) \leq 0$, for all $t, s \in [0, 2\pi]$, then

$$|\widehat{u}(t,\ell)_{\ell\ell}| \leq \int_0^{2\pi} e^{-\mathrm{Re}(q)s} e^{\ell\varphi(t,s)} ds \leq 2\pi e^{|2\pi \mathrm{Re}(q)|},$$

and the sequence $\{\widehat{u}(\,\cdot\,,\ell)\}$ defines a distribution that, by construction, is a solution of $Pu = f$, with f defined in (4.15).

□

Proposition 4.9. *There is no real analytic function defined on* $\mathbb{T}^1 \times \mathbb{S}^3$ *whose sequence of partial Fourier coefficients is given by* (4.16).

Proof. We will prove that the distribution $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$ defined in Proposition 4.8 cannot be a real analytic function by proving that the sequence of numbers $\{\widehat{u}(t^*,\ell)\}_\ell$ does not decay exponentially.

First, observe that for $s \in [0, 2\pi]$, we have

$$\Phi(t^*,s) = \big[B(t^*-s) + K(1-\cos(t^*-s))\big] + i\big[a(0)\sin(t^*-s) - A(t^*-s)\big],$$

in particular, $\Phi(t^*,t^*) = 0$.

Since we are assuming that b is real analytic and changes sign from $-$ to $+$ at $t_0 = 0$, there exists $\delta^* > 0$ such that $B(u) > 0$ for all u such that $0 < |u| < \delta^*$. In particular,

$$B(t^*-s) > 0, \text{ whenever } 0 < |t^*-s| < \delta^*,$$

which implies that

$$\mathrm{Re}(\Phi(t^*,s)) > 0, \text{ in } 0 < |t^*-s| < \delta^*.$$

Note that, since $t^* \in (0, 2\pi)$, we can choose δ^* such that $(t^*-\delta^*, t^*+\delta^*) \subset [0, 2\pi]$.
In the region $|t^*-s| \geq \delta^*$, we have $1 - \cos(t^*-s) \neq 0$. Setting

$$\rho^* = \min\{B(t^*-s); |t^*-s| \geq \delta^*, s \in [0, 2\pi]\},$$

we have

$$\mathrm{Re}(\Phi(t^*,s)) = B(t^*-s) + K(1-\cos(t^*-s)) \geq \rho^* + K(1-\cos(t^*-s)).$$

But

$$\rho^* + K(1-\cos(t^*-s)) > 0 \iff K > \frac{-\rho^*}{1-\cos(t^*-s)}.$$

Therefore, increasing $K > 0$ if necessary, we have that

$$\mathrm{Re}(\Phi(t^*, s)) > 0 \text{ for all } s \in [0, 2\pi], \text{with } s \neq t^*. \tag{4.17}$$

The next step is is to study the asymptotic behavior, as $\ell \to \infty$, of the sequence

$$\widehat{u}(t^*, \ell)_{\ell\ell} = \int_0^{2\pi} e^{-qs} e^{-\ell\Phi(t^*,s)} ds = I_{\ell,\delta} + J_{\ell,\delta}, \tag{4.18}$$

where

$$\begin{aligned} I_{\ell,\delta} &= \int_{R_\delta} e^{-qs} e^{-\ell\Phi(t^*,s)} ds \\ &= \int_{\widetilde{R}_\delta} e^{-q(t^*-\sigma)} e^{-\ell[B(\sigma)+K(1-\cos(\sigma))+i(a(0)\sin(\sigma)-A(\sigma))]} d\sigma, \end{aligned}$$

and

$$\begin{aligned} J_{\ell,\delta} &= \int_{(R_\delta)^c} e^{-qs} e^{-\ell\Phi(t^*,s)} ds \\ &= \int_{(\widetilde{R}_\delta)^c} e^{-q(t^*-\sigma)} e^{-\ell[B(\sigma)+K(1-\cos(\sigma))+i(a(0)\sin(\sigma)-A(\sigma))]} d\sigma, \end{aligned}$$

with $R_\delta = \{s \in [0, 2\pi]; |t^* - s| < \delta\}$ and $\widetilde{R}_\delta = \{\sigma \in [t^* - 2\pi, t^*]; |\sigma| < \delta\}$. Now, if $|\sigma| \geq \delta$ and $\rho = \min_{|\sigma|\geq\delta} B(\sigma)$, then, again increasing $K > 0$ if necessary,

$$B(\sigma) + K(1 - \cos(\sigma)) \geq \rho + K(1 - \cos(\delta)) = C_{K,\delta} > 0$$

therefore

$$|J_{\ell,\delta}| \leq C \int_{(\widetilde{R}_\delta)^c} e^{-\ell(\rho+K(1-\cos(\delta)))} d\sigma \leq \widetilde{C} e^{-\ell C_{K,\delta}}. \tag{4.19}$$

Now let us analyze the integral $I_{\ell,\delta}$. Notice that

$$I_{\ell,\delta} = \int_{\widetilde{R}_\delta} e^{q(t^*-\sigma)} e^{-\ell\phi(\sigma)} d\sigma,$$

where

$$\begin{aligned} \phi(\sigma) &= (B(\sigma) + K(1 - \cos(\sigma))) + i(a(0)\sin(\sigma) - A(\sigma)) \\ &= -iC(\sigma) + K(1 - \cos(\sigma)) + ia(0)\sin(\sigma). \end{aligned}$$

Replacing σ by the complex variable $z = \sigma + i\tau$ in the above expression, we obtain a function $\phi(z)$ holomorphic on some square $\{z = \sigma + i\tau \in \mathbb{C}; |\sigma|, |\tau| < \delta\}$. Thus

$$\phi'(z) = -i(a(z) + ib(z)) + K\sin(z) + ia(0)\cos(z), \text{ and}$$

$$\phi''(z) = -i(a'(z) + ib'(z)) + K\cos(z) - ia(0)\sin(z),$$

and

$$\begin{aligned}
\phi(0) &= -i(A(0) + iB(0)) = 0,\\
\phi'(0) &= -ia(0) + ib(0) + ia(0) = 0, \text{ and}\\
\phi''(0) &= -i(a'(0) + ib'(0)) + K = K - ic'(0) \neq 0, \text{ if } K \text{ is large enough.}
\end{aligned}$$

Since $\phi(z)$ is holomorphic in some small neighborhood of the origin, and $z = 0$ is the only critical point of $\phi(z)$ on the square $\{z \in \mathbb{C}; |z| < \delta\}$, for δ sufficiently small, then the function $\phi(z)$ fits the hypothesis of Theorem 2.8 in [26] and there exists $\varepsilon > 0$ such that

$$I_{\ell,\delta} = \sqrt{2\pi}\frac{1}{\sqrt{2\ell}} + \frac{1}{\varepsilon}e^{-2\ell\varepsilon}, \tag{4.20}$$

for all $\ell \in \frac{1}{2}\mathbb{N}$.

It follows from (4.18), (4.19) and (4.20) that

$$\begin{aligned}
|\widehat{u}(t^*, \ell)| =&|I_{\ell,\delta} + J_{\ell,\delta}|\\
\geq&\sqrt{2\pi}\frac{1}{\sqrt{2\ell}} + \frac{1}{\varepsilon}e^{-2\ell\varepsilon} - 2\pi e^{-\ell C_{K,\delta}}\\
=&O\left(\frac{1}{\sqrt{\ell}}\right),
\end{aligned}$$

when $\ell \to \infty$.

Since $\widehat{u}(t^*, \ell)$ does not decay exponentially, the sequence of functions $\{\widehat{u}(t, \ell)_{mn}\}$ does not correspond to any real analytic function on $\mathbb{T}^1 \times \mathbb{S}^3$, what finishes the proof.

□

Finally, from Propositions 4.6, 4.8 and 4.9 we have the following theorem.

Theorem 4.10. *If* $\operatorname{Im} c(t)$ *changes sign, then* $P = \partial_t + c(t)\partial_0 + q$ *is not (GAH) for all* $q \in \mathbb{C}$.

4.4 Main Theorem and examples

In this section, we present the remaining result that is missing to turn Theorem 4.3 into an equivalence and also give some examples of global analytic hypoelliptic operators.

The next result establishes a relation between the global analytic hypoellipticity of P and P_0 defined in (4.1) and (4.3) respectively.

Theorem 4.11. *If P is (GAH), then P_0 is also (GAH).*

Proof. Suppose that P_0 is not (GAH). By Proposition 3.3, the set $\mathcal{N}_0$ has infinitely many elements or P_0 does not satisfy the (ADC) condition. If $\mathcal{N}_0$ is infinite, then by Proposition 4.1 we have that P is not (GAH) and we are done in this case. So now suppose that $\mathcal{N} = \varnothing$ and P_0 does not satisfy the (ADC) condition. In this case, there exist a constant $B > 0$, a sequence of representations $\{(k_j, \ell_j)\}_j$ in $\mathbb{Z} \times \frac{1}{2}\mathbb{N}$ and indexes $-\ell_j \leq m_j \leq \ell_j$ such that

$$0 < |k_j + c_0 m_j - iq| = |(k_j + \text{Im}(q) + m_j a_0) - i(\text{Re}(q) - m_j b_0)| \leq \frac{1}{j} e^{-B(|k_j| + \ell_j)}$$

for all $j \in \mathbb{N}$. Notice that the set $\{m_j\}$ cannot be bounded because the right hand-side above goes to 0 and the middle term assume a discrete set of values, so it must be the constant sequence equals to 0 for big enough values of j, which contradicts it being positive for all j. So we can assume without loss of generality that $m_j \to \infty$. The convergence implies, in particular, that $\text{Re}(q) - m_j b_0 \to 0$, but since $\{m_j\}$ is unbounded, this implies that $\text{Re}(q) = b_0 = 0$. Therefore $b \equiv 0$ or b changes sign. If $b \equiv 0$, then P is conjugated to P_0 by Proposition 4.5 and then P is not (GAH) since P_0 is not by assumption. If b changes sign, then P is not (GAH) by Theorem 4.10. In any case, we concluded that P is not (GAH). $\square$

Now we can summarize the Theorem 4.3 with the above results turning that theorem into an equivalence.

Theorem 4.12. *The operator $P = \partial_t + c(t)\partial_0 + q$ defined in (4.1) is (GAH) if and only if the three conditions bellow are satisfied:*

1. $\text{Im}\, c(t)$ *does not change sign;*
2. q *satisfies any of the conditions (C1) or (C2);*
3. $P_0 = \partial_t + c_0\partial_0 + q$ *satisfies the (ADC) condition.*

Remark 4.13. *If $b \equiv 0$, in view of Proposition 4.5, we have that P is (GAH) if and only if Conditions 2 and 3 are satisfied. Condition 1 is automatic in this case. Moreover, by Propositions 3.3 and 3.4, Conditions 2 and 3 together are equivalent to P_0 being (GAH), so we could state Theorem 4.12 saying that P is (GAH) if and only if* $\text{Im}\, c(t)$ *does not change sign and P_0 is (GAH).*

Remark 4.14. *We claim that if $b \not\equiv 0$, then Conditions 1 and 2 together imply Condition 3. Observe Condition 2 says that for all $m \in \frac{1}{2}\mathbb{Z}$ and $k \in \mathbb{Z}$ we have*

$$\gamma_{k,m} = k + c_0 m + iq = (k + a_0 m + \text{Im}(q)) + i(mb_0 - \text{Re}(q)) \neq 0.$$

Besides, since $b \not\equiv 0$ and does not change sign, we have $b_0 \neq 0$. Now let us split the proof in some cases. First, suppose that $\text{Re}(q)/b_0 \in \frac{1}{2}\mathbb{Z}$. In this case, we must have $\text{Re}(\gamma_{k,m}) \neq 0$ for $m = \text{Re}(q)/b_0$, which implies $a_0 m + \text{Im}(q) \notin \mathbb{Z}$ and therefore

$$\min_{k \in \mathbb{Z}} \{|k + a_0 m + \text{Im}(q)|\} = C_1 > 0,$$

and so $|\gamma_{k,m}| \geq C_1 \geq C_1 e^{-B(|k|+|m|)}$. Now suppose that $\text{Re}(q)/b_0 \notin \frac{1}{2}\mathbb{Z}$, in particular we have $\text{Re}(q) \neq 0$. Given $m \in \frac{1}{2}\mathbb{Z}$, $m \neq 0$, then $mb_0 - \text{Re}(q) \neq 0$, so

$$\min\left\{|mb_0 - \text{Re}(q)|; m \in \tfrac{1}{2}\mathbb{Z} \setminus \{0\}\right\} = \widetilde{C}_2 > 0,$$

If $m = 0$, then $|b_0 m - \text{Re}(q)| = |\text{Re}(q)| > 0$, so for all k, m we have

$$|\gamma_{k,m}| \geq |b_0 m - \text{Re}(q)| \geq \min\{\widetilde{C}_2, |\text{Re}(q)|\} = C_2 \geq C_2 e^{-B(|k|+|m|)}.$$

In any case, we proved that P_0 satisfies the (ADC) condition.

Example 4.15. *Consider the continued fraction $\alpha = \left[10^{1!}, 10^{2!}, 10^{3!}, \ldots\right]$ and the operator*

$$P_1 = \partial_t + (\alpha + i\sin(t))\partial_0 + i\tfrac{1}{2}.$$

Notice that P_1 satisfies Conditions 2 and 3 of Theorem 4.12 (see Proposition 6.2 of [1]). However, since $b(t) = \sin(t)$ changes sign, we conclude by Theorem 4.12 that P_1 is not (GAH). Consider now the operator

$$P_2 = \partial_t + (\alpha + i(\sin(t) + 2))\partial_0 + \left(1 + i\left(2 - \tfrac{\alpha}{2}\right)\right).$$

The operator P_2 satisfies Condition 1, but does not satisfy Condition 2. Therefore P_2 is not (GAH).

We observe that some operators which are vector fields perturbed by functions can be conjugated to vector fields perturbed by constants. We are going to estate this result in a more general setting than the one in this section.

Proposition 4.16. *Let G be a compact Lie group, $q \in C^\omega(G)$, $q_0 \in \mathbb{C}$ the average of q and let $X \in \mathfrak{g}$. Consider the operators $L = X + q$ and $L_0 = X + q_0$ acting on $\mathcal{D}'(G)$. Assume that there exists a function $Q \in C^\omega(G)$ such that $X(Q) = q - q_0$. Then, $\Phi : \mathcal{D}'(G) \to \mathcal{D}'(G)$ given by $\Phi(u) = e^Q \cdot u$ is continuous, linear, restricts to $\Phi : C^\omega(G) \to C^\omega(G)$ and satisfy*

$$\Phi \circ L = L_0 \circ \Phi.$$

In particular, L is (GAH) if and only if L_0 is (GAH).

Proof. Using the power series of the exponential, we can easily prove that $X(Q) = e^Q X(Q)$. Thus, given $u \in \mathcal{D}'(G)$ we have

$$\begin{aligned} L_0(\Phi(u)) &= (X + q_0)(e^Q u) \\ &= X(e^Q u) + q_0 e^Q u \\ &= X(e^Q)u + e^Q X(u) + q_0 e^Q u \\ &= e^Q X(Q)u + e^Q X(u) + q_0 e^Q u \\ &= e^Q \cdot [qu - q_0 u + X(u) + q_0 u] \\ &= \Phi(L(u)). \end{aligned}$$

□

In the particular case where

$$P = \partial_t + c(t)\partial_0 + q(t,x), \tag{4.21}$$

with $c \in C^\omega(\mathbb{T}^1)$ and $q \in C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$, we are led to consider also the constant-coefficient operator

$$P_{00} \doteq \partial_t + c_0\partial_0 + q_0, \tag{4.22}$$

where

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} c(s)ds \ \text{ and } \ q_0 = \frac{1}{2\pi}\int_{\mathbb{S}^3}\int_0^{2\pi} q(s,x)dsdx.$$

In this situation, we have the following result.

Corollary 4.17. *Given $q \in C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$, assume that there is $Q \in C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$ such that*

$$(\partial_t + c(t)\partial_0)Q = q - q_0.$$

Then we have

$$P \circ e^{-Q} = e^{-Q} \circ P_{00}.$$

in both $\mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3)$ and in $C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$. Hence, P is (GAH) if and only if P_{00} is (GAH).

Example 4.18. *Consider the function* $Q : SU(2) \to \mathbb{C}$ *given by* $Q(a,b) = -2a\bar{b}$*, where* $(a,b) \in \mathbb{C}^2$ *with* $|a|^2 + |b|^2 = 1$*. In Euler-angles coordinates we have* $Q(\phi, \theta, \psi) = ie^{i\psi}\sin(\theta)$*, hence* $\partial_0 Q = i\frac{\partial}{\partial\psi}Q = -Q$*. If we define* $q : \mathbb{T}^1 \times \mathbb{S}^3 \to \mathbb{C}$ *by* $q(t,a,b) = 2(\alpha + i\sin(t))a\bar{b} + i/2$*, then* $q_0 = i/2$ *and* $(\partial_t + (\alpha + i\sin(t))\partial_0)Q = q - q_0$*. In this way, the operator*

$$P = \partial_t + (\alpha + i\sin(t))\partial_0 + q$$

is conjugated with the operator P_1 *in example 4.15 by Proposition 4.17, in particular,* P *is not (GAH).*

Example 4.19. *Let* $n \in \mathbb{Z} \setminus \{0\}$*,* $a \in C^\omega(\mathbb{T}^1)$*,* $c(t) = a(t) + i(\sin(t) + n)$*, and* $q : \mathbb{T}^1 \times \mathbb{S}^3 \to \mathbb{C}$ *given by* $q(t,a,b) = 2c(t)a\bar{b} + \sqrt{2}$*. Hence,* $q_0 = \sqrt{2}$ *and*

$$(\partial_t + c(t)\partial_0)Q = q - q_0,$$

with Q *as in Example 4.18. Therefore, by Proposition 4.17, the operator* $P = \partial_t + c(t)\partial_0 + q$ *is conjugated with the operator* $\partial_t + c(t)\partial_0 + \sqrt{2}$*, which is (GAH) as we saw in Example 4.4. In particular,* P *is also (GAH).*

Chapter 5

Evolution operators with more variables

In view of the works [4] and [5], a natural extension of the results obtained in the previous chapter is to consider operators of the form

$$P = \partial_t + c^1(t)\partial_{0,1} + ... + c^N(t)\partial_{0,N} + q.$$

defined on $\mathbb{T}^1 \times \mathbb{S}^3 \times ... \times \mathbb{S}^3$, with N copies of the 3-sphere. Here, each operator $\partial_{0,j}$ is the neutral operator ∂_0 on the j-th 3-sphere factor, $c^1, ..., c^N \in C^\omega(\mathbb{T}^1)$ and $q \in \mathbb{C}$.

We will write $c^j = a^j + ib^j$, $j = 1, ..., N$, with a^j, b^j being real analytic and real-valued functions and

$$c_0^j = a_0^j + ib_0^j = (2\pi)^{-1} \int_0^{2\pi} c^j(\sigma)d\sigma.$$

Recall that a partial Fourier coefficient with respect to a $\mathbb{S}^3$ factor is a matrix, so if we take successive partial Fourier Series with respect to each 3-spheres factors, we will have to take care about a lot of indexes. However, since the order with we take partial Fourier Series does not matter, we will use the following vector notation.

Consider the vector valued functions given by

$$a = (a^1, ..., a^N), b = (b^1, ..., b^N), c = a + ib$$

and their respective average vectors $a_0 = (a_0^1, ..., a_0^N), b_0 = (b_0^1, ..., b_0^N)$ and $c_0 = a_0 + ib_0$.

We also will use the analogue notation for periodic primitives,

$$\mathscr{C}_j(t) = \int_0^t c^j(\sigma)d\sigma - c_0^j t.$$

If $u \in \mathcal{D}'(\mathbb{T}^1 \times (\mathbb{S}^3)^N)$, $\ell = (\ell^1, ..., \ell^N) \in \frac{1}{2}\mathbb{N}_0^N$, and $m^j, n^j \in \frac{1}{2}\mathbb{Z}$ are such that $-\ell^j \leq m^j, n^j \leq \ell^j$, then we will write

$$\widehat{u}(t, \ell)_{(m^1,n^1)...(m^N,n^N)} =: \widehat{u}(t, \ell)_{mn},$$

where $m = (m^1, ..., m^N)$ and $n = (n^1, ..., n^N)$.

When $\ell \in \frac{1}{2}\mathbb{N}_0^N$ and $m \in \frac{1}{2}\mathbb{Z}^N$ the notation $-\ell \leq m \leq \ell$ means that, for every $j = 1, 2, ..., N$ we have $-\ell^j \leq m^j \leq \ell^j$. And when we say "let $\ell \in \frac{1}{2}\mathbb{N}_0^N$ and m, n indexes such that $-\ell \leq m, n \leq \ell$" it is assumed that $m, n \in \frac{1}{2}\mathbb{Z}^N$.

Given $z = (z^1, ..., z^N), w = (w^1, ..., w^N) \in \mathbb{C}^N$, we will write

$$z \cdot w = \overline{z^1}w^1 + ... + \overline{z^N}w^N$$

for the standard hermitian product on $\mathbb{C}^N$ and

$$|z| = |z^1| + ... + |z^N|.$$

With this notation, if $u \in \mathcal{D}'(\mathbb{T}^1 \times (\mathbb{S}^3)^N)$, $\tau \in \mathbb{Z}$, $\ell \in \frac{1}{2}\mathbb{N}_0^N$, and m, n are indexes such that $-\ell \leq m, n \leq \ell$, then

$$\widehat{Pu}(\tau, \ell)_{mn} = [\tau + im \cdot c_0 + q]\, \widehat{u}(\tau, \ell)_{mn},$$

so the notation stays similar to the case of only one 3-sphere factor. Therefore it is natural to consider the analogous set of "null frequencies". Define

$$\mathcal{N}_0 = \left\{ \ell \in \frac{1}{2}\mathbb{N}_0^N;\ \text{there exists an index } -\ell \leq m \leq \ell \text{ such that } m \cdot c_0 - iq \in \mathbb{Z} \right\}$$

Proposition 5.1. *If P is (GAH), then $\mathcal{N}_0$ is finite.*

Proof. Suppose by contradiction that $\mathcal{N}_0$ has infinitely many elements. In this case, there are sequences $\{(k_j, \ell_j) \in \mathbb{Z} \times \frac{1}{2}\mathbb{N}_0^N\}_{j \in \mathbb{N}}$ and $\{m_j\}_{j \in \mathbb{N}}$, with $-\ell_j \leq m_j \leq \ell_j$, such that

$$\tau_j + c_0 m_j - iq = 0,\ j \in \mathbb{N}. \tag{5.1}$$

For each $j \in \mathbb{N}$, let $t_j \in [0, 2\pi]$ and $M_j \in \mathbb{R}$ such that

$$M_j = \int_0^{t_j} \big[\mathrm{Re}(q) - m_j b(\sigma)\big] d\sigma = \max_{0 \leq t \leq 2\pi} \int_0^t \big[\mathrm{Re}(q) - m_j b(\sigma)\big] d\sigma,$$

and consider the sequence

$$\widehat{u}(t, \ell)_{mn} = \begin{cases} \exp\left\{ \displaystyle\int_0^t \big[i m_j c(\sigma) + q\big] d\sigma - M_j \right\}, & \text{if } \ell = \ell_j \text{ and } m = n = m_j \\ 0 & \text{otherwise.} \end{cases}$$

Since $im_j c_0 + q \in i\mathbb{Z}$, for all $j \in \mathbb{N}$, all the partial Fourier coefficients $\widehat{u}(\cdot, \ell) \in C^\infty(\mathbb{T}^1)$ and

$$|\widehat{u}(t, \ell_j)_{m_j m_j}| = \exp\left\{ \int_0^t \big[\mathrm{Re}(q) - m_j b(\sigma)\big] - M_j \right\} \leq 1,\ \text{for all } t \in [0, 2\pi],$$

therefore the sequence $\{\widehat{u}(\cdot,\ell); \ell \in \frac{1}{2}\mathbb{N}\}$ correspond to a distribution $u \in \mathcal{D}'(\mathbb{T}^1 \times (\mathbb{S}^3)^N)$, moreover

$$\widehat{u}(t_j,\ell_j)_{m_j m_j} = 1,\ j \in \mathbb{N}.$$

Now note that set $\{\ell_j\}_{j\in\mathbb{N}}$ cannot be bounded. Indeed, if it were bounded, by (5.1), the set $\{\tau_j\}_{j\in\mathbb{N}}$ would also be bounded and, consequently, the set $\mathcal{N}_0$ would be finite (because $\mathcal{N}_0$ is discrete). Hence, $\{\ell_j\}_{j\in\mathbb{N}}$ has infinitely many elements and $u \notin C^\infty(\mathbb{T}^1 \times (\mathbb{S}^3)^N)$. Since

$$[\partial_t + im_j c(t) + q]\widehat{u}(t,\ell_j)_{m_j m_j} = 0,\ j \in \mathbb{N},$$

then $Pu = 0$ and P is not (GAH).

□

Note that, as in Remark 3.8, if the set $\mathcal{N}_0$ is finite then this implies that $\mathcal{N}_0 = \varnothing$, by the same reason we explored in that remark.

We consider the constant coefficient operator P_0 defined by

$$P_0 = \partial_t + c_0^1\partial_{0,1} + ... + c_0^N\partial_{0,N} + q.$$

Recall that by Theorem 3.5 and Proposition 3.6 in the case $G_2 = (\mathbb{S}^3)^N$, the operator P_0 is (GAH) if and only if for all $B > 0$, there exists $K > 0$ such that

$$|1 - e^{\pm 2\pi(im\cdot c_0 + q)}| \geq K e^{-B|\ell|}$$

for all $\ell \in \frac{1}{2}\mathbb{N}_0^N$ and indexes $-\ell \leq m \leq \ell$.

Remark 5.2. *If b^j changes sign, for some $j = 1,\dots,N$, then the operator P is not (GAH). Indeed, if some b^j changes sign, then $P_j = \partial_t + c^j\partial_{0j} + q$ is not (GAH) in $\mathbb{T}^1 \times \mathbb{S}^3$, therefore there is a distribution $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3) \setminus C^\infty(\mathbb{T}^1 \times \mathbb{S}^3)$ such that $P_j u \in C^\omega(\mathbb{T}^1 \times \mathbb{S}^3)$.*
But then we can think u as a distribution on the whole $\mathbb{T}^1 \times (\mathbb{S}^3)^N$ by taking its tensor product with constant function equal to 1 in the remaining variables, so that $P(u) = P_j(u)$, from where can we conclude that P is not (GAH).

In view of the last proposition and remark, from now on we will assume that:

1. the functions b^j do not change sign, for all $j = 1, ..., N$; and

2. $\mathcal{N}_0$ is finite.

Inspired in [4], we now present a third necessary condition for the global analytic hypoellipticity of P.

Proposition 5.3. *If the set $\{b^1, ..., b^N\}$ is linearly independent, then P is not (GAH).*

Proof. If $\{b^1, ..., b^N\}$ is linearly independent, then, in particular, $\{b^1, b^2\}$ is linearly independent. Let us prove that $P = \partial_t + c^1\partial_{01} + c^2\partial_{02} + q$ is not (GAH) in $\mathbb{T}^1 \times \mathbb{S}^3 \times \mathbb{S}^3$, which implies that the whole operator P is not be (GAH) (again by taking tensor products).

Since $\{b^1, b^2\}$ is linearly independent, by Lemma 3.1 in [5], there exist integer numbers $r, s \in \mathbb{Z}$ such that the function $\widetilde{b} := rb^1 + sb^2$ changes sign. We also set $\widetilde{c} = rc^1 + sc^2$ and $Pu = f$, where $u \in \mathcal{D}'(\mathbb{T}^1 \times \mathbb{S}^3 \times \mathbb{S}^3)$.

Suppose for example that $r < 0$ and $s > 0$. In this case, consider the frequencies of the form $(\ell_1, \ell_2) = (-kr, ks)$, with $k \in \mathbb{N}$ and indexes $m = (kr, kr), n = (ks, ks)$. Since we are supposing that $\mathcal{N}_0 = \varnothing$, the unique solution of each equation

$$\widehat{Pu}(t, -kr, ks)_{(kr,kr)(ks,ks)} = \widehat{f}(t, -kr, ks)_{(kr,kr)(ks,ks)}$$

is given by

$$\widehat{u}(t, -kr, ks)_{(kr,kr)(ks,ks)} = d_k^{-1} \int_0^{2\pi} e^{-qx} \exp\left(-ik \int_{t-x}^{t} \widetilde{c}(\sigma)d\sigma\right) \widehat{f}(t-x, -kr, ks)_{(kr,kr)(ks,ks)}dx,$$

where $d_k = 1 - \exp(-2\pi(ikc_0 + q))$.

Note that now we have exactly the same situation as when we built a singular solution for the case of only one $\mathbb{S}^3$-factor. In fact, the expression of the solution involves a real analytic function $\widetilde{c}$ such that its imaginary part changes sign. So we just mimic the definition of the analytic function f in (4.15) and follow the same lines of that proof to see that for this f there is no real analytic function as solution for the equation $Pu = f$.

$\square$

Up to this point we have already proved that if P is (GAH), then: all the functions b_j, $j = 1, ..., N$, cannot change sign, the set $\{b_1, ..., b_N\}$ is linearly dependent, and $\mathcal{N}_0 = \varnothing$.

Let us prove now that these conditions are also sufficient for P to be (GAH), if we further add the assumption that P_0 satisfies the (ADC) condition, which is expected because this is true when $N = 1$.

Recall that, by Proposition 3.5, a constant coefficient operator is (GAH) if and only if it satisfies the (ADC)-condition and $\mathcal{N}_0 = \varnothing$, therefore we have the following result.

Proposition 5.4. *If*

- P_0 *is (GAH);*
- b^j *does not change sign, for all* $j = 1, ..., N$*;*
- $\{b^1, ..., b^N\}$ *is linearly dependent,*

then the operator P *is (GAH).*

Proof. Let $f \in C^\omega(\mathbb{T}^1 \times (\mathbb{S}^3)^N)$ and suppose that there is a distribution u such that $Pu = f$. We will prove that $u \in C^\omega(\mathbb{T}^1 \times (\mathbb{S}^3)^N)$.
If $\ell \in \frac{1}{2}\mathbb{N}_0^N$ and $-\ell \leq m, n \leq \ell$, then $\widehat{Pu}(t, \ell)_{(m,n)} = \widehat{f}(t, \ell)_{(m,n)}$ is equivalent to

$$[\partial_t + im \cdot c(t) + q] \exp\left(i \sum_{j=1}^{N} m^j \mathscr{C}_j(t)\right) \widehat{u}(t, \ell)_{(m,n)} = \exp\left(i \sum_{j=1}^{N} m^j \mathscr{C}_j(t)\right) \widehat{f}(t, \ell)_{(m,n)}.$$

It follows from the hypothesis $\mathcal{N}_0 = \varnothing$ that for each j $(1 \leq j \leq N)$ the equation above has only one vectorial periodic solution, which can be written in the two following equivalent ways:

$$\widehat{u}(t, \ell)_{(m,n)} = d_m^{-1} \int_0^{2\pi} e^{-qx} e^{-i \sum_j m^j \int_{t-x}^{t} c^j(\sigma) d\sigma} \widehat{f}(t - x, \ell)_{(m,n)} dx, \tag{5.2}$$

where $d_m = 1 - \exp(-2\pi(m \cdot c_0 + q))$, or

$$\widehat{u}(t, \ell)_{(m,n)} = \widetilde{d}_m^{-1} \int_0^{2\pi} e^{qx} e^{i \sum_j m^j \int_t^{t+x} c^j(\sigma) d\sigma} \widehat{f}(t + x, \ell)_{(m,n)} dx, \tag{5.3}$$

where $\widetilde{d}_m = \exp((2\pi(im \cdot c_0 + q))) - 1$.
Since we are supposing that the set $\{b^1, ..., b^N\}$ is linearly dependent, then there exist real numbers $\mu^1, ..., \mu^N$ and $b \in C^\omega(\mathbb{T}^1)$ such that $b^j = \mu^j b$ for all $j = 1, ..., N$. And, moreover, we are supposing that each b^j does not change sign, so b also cannot. Hence, the formulas (5.2) and (5.3) become, respectively,

$$\widehat{u}(t, \ell)_{(m,n)} = d_m^{-1} \int_0^{2\pi} e^{-qx} e^{-i \sum_j m^j \int_{t-x}^{t} a^j(\sigma) d\sigma} e^{\sum_j m^j \mu^j \int_{t-x}^{t} b(\sigma) d\sigma} \widehat{f}(t - x, \ell)_{(m,n)} dx, \tag{5.4}$$

and

$$\widehat{u}(t, \ell)_{(m,n)} = \widetilde{d}_m^{-1} \int_0^{2\pi} e^{qx} e^{i \sum_j m^j \int_t^{t+x} a^j(\sigma) d\sigma} e^{-\sum_j m^j \mu^j \int_t^{t+x} b(\sigma) d\sigma} \widehat{f}(t + x, \ell)_{(m,n)} dx. \tag{5.5}$$

Now, if $f \in C^\omega(\mathbb{T}^1 \times (\mathbb{S}^3)^N)$, there exist $B > 0$ and $M > 0$ such that

$$|\widehat{f}(t, \ell)_{(m,n)}| \leq M e^{-B\ell}, \quad \text{for all } \ell, m, n.$$

Since we are supposing that P_0 satisfies the (ADC)-condition, choose any $0 < \widetilde{B} < B$ (so that $B - \widetilde{B} > 0$), and for this $\widetilde{B}$ there exists $K > 0$ such that

$$\min\{|d_m|, |\widetilde{d}_m|\}| \geq Ke^{-\widetilde{B}\ell}, \text{ for all } \ell, m.$$

Note that also $|e^{qx}| \leq e^{2\pi|q|} = C$ for $x \in [0, 2\pi]$.

Suppose without loss of generality that $b \leq 0$, so that $\max\left\{\int_{t-x}^{t} b, \int_{t}^{t+x} b\right\} \leq 0$ for all real numbers $t, x \in [0, 2\pi]$.

In the set of indexes m such that $\sum_j m^j \mu^j \geq 0$ we estimate $|\widehat{u}(t, \ell)_{(m,n)}|$ using (5.2) to obtain a constant $\widetilde{C} > 0$ such that

$$|\widehat{u}(t, \ell)_{(m,n)}| \leq \widetilde{C}e^{-(B-\widetilde{B})\ell}, \text{ for all } \ell, m, n.$$

Similarly, in the set of indexes such that $\sum_j m^j \mu^j \leq 0$ we use (5.3) to estimate $|\widehat{u}(t, \ell)_{(m,n)}|$ and obtain an analogous result, so that $u \in C^\omega(\mathbb{T}^1 \times (\mathbb{S}^3)^N)$ and P is (GAH).

$\square$

References

[1] Alexandre Arias Junior, Alexandre Kirilov, and Cleber de Medeira. Global Gevrey hypoellipticity on the torus for a class of systems of complex vector fields. *J. Math. Anal. Appl.*, 474(1):712–732, 2019.

[2] A. P. Bergamasco. Remarks about global analytic hypoellipticity. *Trans. Amer. Math. Soc.*, 351(10):4113–4126, 1999.

[3] Adalberto P. Bergamasco. Perturbations of globally hypoelliptic operators. *J. Differential Equations*, 114(2):513–526, 1994.

[4] Adalberto P. Bergamasco, Paulo L. Dattori da Silva, and Rafael B. Gonzalez. Existence and regularity of periodic solutions to certain first-order partial differential equations. *J. Fourier Anal. Appl.*, 23(1):65–90, 2017.

[5] Adalberto P. Bergamasco, Paulo L. Dattori da Silva, Rafael B. Gonzalez, and Alexandre Kirilov. Global solvability and global hypoellipticity for a class of complex vector fields on the 3-torus. *J. Pseudo-Differ. Oper. Appl.*, 6(3):341–360, 2015.

[6] Adalberto P. Bergamasco, Wagner V. L. Nunes, and Sérgio Luís Zani. Global analytic hypoellipticity and pseudoperiodic functions. volume 18, pages 43–57. 2000. VI Workshop on Partial Differential Equations, Part I (Rio de Janeiro, 1999).

[7] Adalberto P. Bergamasco and Sérgio Luís Zani. Prescribing analytic singularities for solutions of a class of vector fields on the torus. *Trans. Amer. Math. Soc.*, 357(10):4159–4174, 2005.

[8] Adalberto P. Bergamasco and Sérgio Luís Zani. Globally analytic hypoelliptic vector fields on compact surfaces. *Proc. Amer. Math. Soc.*, 136(4):1305–1310, 2008.

[9] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[10] Aparajita Dasgupta and Michael Ruzhansky. Eigenfunction expansions of ultradifferentiable functions and ultradistributions. *Trans. Amer. Math. Soc.*, 368(12):8481–8498, 2016.

[11] F. de Ávila Silva, T. Gramchev, and A. Kirilov. Global hypoellipticity for first-order operators on closed smooth manifolds. *J. Anal. Math.*, 135(2):527–573, 2018.

[12] Fernando de Ávila Silva, Rafael Borro Gonzalez, Alexandre Kirilov, and Cleber de Medeira. Global hypoellipticity for a class of pseudo-differential operators on the torus. *J. Fourier Anal. Appl.*, 25(4):1717–1758, 2019.

[13] Fernando de Ávila Silva and Alexandre Kirilov. Perturbations of globally hypoelliptic operators on closed manifolds. *J. Spectr. Theory*, 9(3):825–855, 2019.

[14] M. Ferrer, S. Hernández, and V. Uspenskij. Precompact groups and property (T). *J. Math. Anal. Appl.*, 404(2):221–230, 2013.

[15] Veronique Fischer and Michael Ruzhansky. *Quantization on nilpotent Lie groups*, volume 314 of *Progress in Mathematics*. Birkhäuser/Springer, [Cham], 2016.

[16] G. Forni. On the Greenfield-Wallach and Katok conjectures in dimension three. In *Geometric and probabilistic structures in dynamics*, volume 469 of *Contemp. Math.*, pages 197–213. Amer. Math. Soc., Providence, RI, 2008.

[17] S. J. Greenfield and N. R. Wallach. Global hypoellipticity and Liouville numbers. *Proc. Amer. Math. Soc.*, 31:112–114, 1972.

[18] S. J. Greenfield and N. R. Wallach. Globally hypoelliptic vector fields. *Topology*, 12:247–254, 1973.

[19] S. Helgason. *Groups & geometric analysis: Radon Transforms, Invariant Differential Operators and Spherical Functions*, volume 1 of *ISSN*. Elsevier Science, 1984.

[20] J. Hounie. Globally hypoelliptic and globally solvable first-order evolution equations. *Trans. Amer. Math. Soc.*, 252:233–248, 1979.

[21] J. Hounie. Globally hypoelliptic vector fields on compact surfaces. *Comm. Partial Differential Equations*, 7(4):343–370, 1982.

[22] A. Kirilov, W. A. A. de Moraes, and M. Ruzhansky. Global properties on compact Lie groups in Komatsu classes II: normal form. *arXiv e-prints*, page arXiv:1911.02486 [math.AP], Nov 2019.

[23] A. Kirilov, W. A. A. de Moraes, and M. Ruzhansky. Partial Fourier series on compact Lie groups. *arXiv e-prints*, page arXiv:1909.12824 [math.AP], Sep 2019.

[24] G. Petronilho. Global hypoellipticity, global solvability and normal form for a class of real vector fields on a torus and application. *Trans. Amer. Math. Soc.*, 363(12):6337–6349, 2011.

[25] M. Ruzhansky and V. Turunen. *Pseudo-differential operators and symmetries*, volume 2 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010. Background analysis and advanced topics.

[26] Johannes Sjöstrand. Singularités analytiques microlocales. In *Astérisque, 95*, volume 95 of *Astérisque*, pages 1–166. Soc. Math. France, Paris, 1982.